

PRACTICE FINAL EXAMINATION - SOLUTION

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. Problems 9 and 10 are optional; attempt them if you wish to improve your midterm examination score.

1. State the matrix product to obtain 3 linear combinations of vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

with scaling coefficients $(\alpha_1, \beta_1) = (1, 1)$, $(\alpha_2, \beta_2) = (-1, 1)$, $(\alpha_3, \beta_3) = (1, -1)$.

Solution. The matrix product is $\mathbf{C} = \mathbf{A}\mathbf{B}$, $\mathbf{C} \in \mathbb{R}^{3 \times 3}$ (3 linear combinations, each with 3 components), $\mathbf{A} = [\mathbf{u} \ \mathbf{v}] \in \mathbb{R}^{3 \times 2}$ (vectors entering into linear combination), $\mathbf{B} \in \mathbb{R}^{2 \times 3}$ (scaling coefficients of each linear combination)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

2. Orthonormalize the vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Solution. Note that $\mathbf{u}^T \mathbf{v} = \mathbf{u}^T \mathbf{w} = 0$, such that these vector pairs are already orthogonal. Also note that the vector $\mathbf{e}_2 = [0 \ 1 \ 0]^T$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. Scale vectors to have unit norm

$$\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3] = [\mathbf{u}/\sqrt{2} \ \mathbf{v}/\sqrt{2} \ \mathbf{e}_2].$$

3. For $x, y \in \mathbb{R}$, expansion of $(x - y)^3$ leads to $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$. Find the corresponding expansion of $(\mathbf{A} - \mathbf{B})^3$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times m}$.

Solution. Compute, recalling that matrix multiplication is not commutative

$$(\mathbf{A} - \mathbf{B})^3 = (\mathbf{A} - \mathbf{B})(\mathbf{A}^2 - \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A} + \mathbf{B}^2) = \mathbf{A}^3 - \mathbf{A}^2\mathbf{B} - \mathbf{A}\mathbf{B}\mathbf{A} + \mathbf{A}\mathbf{B}^2 - \mathbf{B}\mathbf{A}^2 + \mathbf{B}\mathbf{A}\mathbf{B} + \mathbf{B}^2 - \mathbf{B}^3.$$

4. Find the projection of $\mathbf{b}$ onto $C(\mathbf{A})$ for

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution. With $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2]$ note that $\mathbf{b} = 2\mathbf{a}_1 + \mathbf{a}_2$, hence $\mathbf{b} \in C(\mathbf{A})$, and the projection of $\mathbf{b}$ onto $C(\mathbf{A})$ is $\mathbf{b}$ itself.

Alternatively, orthonormalize $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2]$ to obtain

$$\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2] = [\mathbf{a}_1/\sqrt{3} \ \mathbf{a}_2/\sqrt{2}].$$

The projection onto $C(\mathbf{A})$ is

$$\begin{aligned} \mathbf{c} = \mathbf{Q}\mathbf{Q}^T\mathbf{b} &= \mathbf{Q}(\mathbf{Q}^T\mathbf{b}) = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \left(\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \Rightarrow \\ \mathbf{c} &= \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2\sqrt{3} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

5. Find the LU decomposition of

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}.$$

Solution. Carry out reduction to upper triangular form, noting multipliers used in the process

$$\mathbf{L}_1 \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}.$$

$$\mathbf{L}_2 \mathbf{L}_1 \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{U}.$$

Find $\mathbf{A} = \mathbf{L}_1^{-1} \mathbf{L}_2^{-1} \mathbf{U} = \mathbf{L} \mathbf{U}$. Compute

$$\mathbf{L} = \mathbf{L}_1^{-1} \mathbf{L}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Verify

$$\mathbf{L} \mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{bmatrix}. \checkmark$$

6. State the eigenvalues and eigenvectors of $\mathbf{R} \in \mathbb{R}^{2 \times 2}$, the matrix describing reflection across the vector $\mathbf{w} = [1 \ 2]^T$.

Solution. From eigenvalue relation $\mathbf{R}\mathbf{x} = \lambda\mathbf{x}$ note that directions not changed by reflection are along $\mathbf{w}$ and orthogonal to $\mathbf{w}$. Deduce

$$\mathbf{x}_1 = \mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda_1 = 1$$

$$\mathbf{x}_2 = \mathbf{w} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \lambda_2 = -1.$$

7. Compute the eigendecomposition of

$$\mathbf{A} = \begin{bmatrix} 5/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 5/2 \end{bmatrix}.$$

Solution. Eigenvalue problem is $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$. Observe that $\mathbf{x}_2 = \mathbf{e}_2 = [0 \ 1 \ 0]^T$, $\lambda_2 = 1$ is an eigenvector, value pair. Compute characteristic polynomial by

$$p(\lambda) = \det(\lambda \mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda - 5/2 & 0 & -1/2 \\ 0 & \lambda - 1 & 0 \\ -1/2 & 0 & \lambda - 5/2 \end{vmatrix} = (\lambda - 1) \begin{vmatrix} \lambda - 5/2 & -1/2 \\ -1/2 & \lambda - 5/2 \end{vmatrix} \Rightarrow$$

$$p(\lambda) = (\lambda - 1) \left(\lambda^2 - 5\lambda + \left(\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \right) = (\lambda - 1)(\lambda^2 - 5\lambda + 6) = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

The other eigenvalues are $\lambda_1=2$, $\lambda_3=3$. Find eigenvectors by computing bases for eigenspaces $N(\mathbf{A} - \lambda_1 \mathbf{I})$ and $N(\mathbf{A} - \lambda_3 \mathbf{I})$.

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\mathbf{A} - \lambda_3 \mathbf{I} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \end{bmatrix} \sim \begin{bmatrix} -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

8. Find the SVD of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution. Matrix has orthogonal columns that are not of unit norm. Construct SVD as

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

9. Find the matrix of the reflection of $\mathbb{R}^2$ vectors across the vector $\mathbf{u} = [1 \ 2]^T$.

Solution. Let $\mathbf{w}$ be the reflection of $\mathbf{v}$ across $\mathbf{u}$, $\mathbf{w} = \mathbf{R}\mathbf{v}$. Let

$$\mathbf{q} = \frac{1}{\|\mathbf{u}\|} \mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

The projection of $\mathbf{v}$ onto the direction of $\mathbf{u}$ is $\mathbf{z} = \mathbf{q}\mathbf{q}^T \mathbf{v}$. The travel from $\mathbf{v}$ to $\mathbf{z}$ is $\mathbf{z} - \mathbf{v}$

$$\mathbf{z} = \mathbf{v} + (\mathbf{z} - \mathbf{v}).$$

The reflection is obtained by doubling the travel distance

$$\mathbf{w} = \mathbf{v} + 2(\mathbf{z} - \mathbf{v}) = 2\mathbf{z} - \mathbf{v} = (2\mathbf{q}\mathbf{q}^T - \mathbf{I})\mathbf{v}.$$

Deduce that the reflection matrix is

$$\mathbf{R} = 2\mathbf{q}\mathbf{q}^T - \mathbf{I}.$$

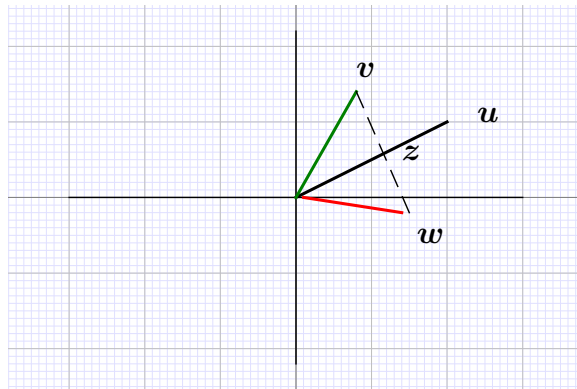


Figure 1.

10. Find bases for the four fundamental spaces of

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Solution. With $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2] \in \mathbb{R}^{3 \times 2}$, the bases are:

$$C(\mathbf{A}): \{\mathbf{a}_1, \mathbf{a}_2\}$$

$$N(\mathbf{A}^T): \{\mathbf{e}_2\} \text{ since } \mathbf{e}_2 \text{ is orthogonal to both } \mathbf{a}_1 \text{ and } \mathbf{a}_2$$

$$C(\mathbf{A}^T): \{\mathbf{a}_1^T, \mathbf{a}_2^T\}$$

$$N(\mathbf{A}): \{\mathbf{0}\}$$