Final Examination: Application

Solve the following problems (5 course points each). Present a brief motivation of your method of solution. This is an open-book test, and you are free to consult the textbook. Your submission must however reflect your individual effort with no aid from any other person. Draft your solution in TeXmacs in this file. Spaces for your solution have been provided in this file for text, formulas, figures, Octave commands. If Octave does not work within TeXmacs, verify your commands in the stand-alone Octave application and paste the commands into the appropriate spaces in this file without executing them. Upload your answer into Sakai both as a TeXmacs, and pdf. Allow at least 10 minutes before the submission cut-off time to ensure you can upload your file.

Problem 1

Data $\mathcal{D}=\{(t_j,x_j), j=1,2,\ldots,m\}$ can be represented in multiple ways. The course desribed the least squares solution to representing the data as a polynomial, for instance a quadratic polynomial $p(t)=c_0+c_1t+c_2t^2$ the coefficients of which are found by solving

$$\min_{\boldsymbol{a}\in\mathbb{R}^3}\|\boldsymbol{x}-\boldsymbol{A}\boldsymbol{c}\|, \boldsymbol{c}=\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}, \boldsymbol{x}=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_m \end{bmatrix}, \boldsymbol{t}^k=\begin{bmatrix} t_1^k\\ t_2^k\\ \vdots\\ t_m^k \end{bmatrix}, \boldsymbol{A}=\begin{bmatrix} \boldsymbol{t}^0 & \boldsymbol{t}^1 & \boldsymbol{t}^2 \end{bmatrix}. \tag{1}$$

Notice that $p(t)$ is a linear combination of $\{1,t,t^2\}$ with scaling coefficients c_0, c_1, c_2, and recall that $t_i^0=1$.

1. Consider another representation of the data as a trigonometric polynomial $q(t)=a_0+a_1\sin(t)+a_2\cos(t)$. State the least squares problem by modifying (1) to reflect the new representation.

   Solution.

   (2)

2. For $m=100$, $t_j=2\pi j/m$ for $j=1,2,\ldots,m$, arbitrarily choose some values for $a_0, a_1, a_2\in[-1,1]$, and construct data vectors $\boldsymbol{t},\boldsymbol{x}\in\mathbb{R}^m$ in Octave.

   Solution.

   ```
   octave]
   ```

3. Solve in Octave the least squares problem you stated in point 1. Do you recover the coefficients a_0, a_1, a_2 you chose?

   ```
   octave]
   ```

4. Perturb the data to mimic measurement noise $\boldsymbol{y}_s=\boldsymbol{x}+s\boldsymbol{r}$, where $\boldsymbol{r}$ is a vector of random numbers in the interval $[-1,1]$, scaled by $s=1$. Solve the least squares problem for the new, noisy data to obtain the perturbed coefficients $\tilde{\boldsymbol{a}}$.

   ```
   octave]
   ```

5. Write an Octave loop over the scaling coefficient values $s=1,2,\ldots,n$ with $n=10$, and compute the norm of the change in the coefficients $e_s=\|\tilde{\boldsymbol{a}}-\boldsymbol{a}\|$ for each s value. Construct a plot of (s,e_s) and comment on the effect of the magnitude of the noise as measured by s upon recovery of the exact coefficients $\boldsymbol{a}$.

   ```
   octave]
   ```

Figure 1. Place your plot of (s,e_s) here using menu item Insert->Image->Insert image …

Problem 2

Continuing the above, suppose the measurement noise is modulated in time,

$$\boldsymbol{y}_s = \boldsymbol{x} + s\boldsymbol{r}\sin(\boldsymbol{t}) + 4s\boldsymbol{r}\sin(2\boldsymbol{t}). \tag{3}$$

Investigate now the utility of the singular value decomposition to gain insight into the data.

1. Construct a data matrix of the modulated noise measurements specified in formula (3)

$$\boldsymbol{Y} = [\ \boldsymbol{y}_1\ \ \boldsymbol{y}_2\ \ \dots\ \ \boldsymbol{y}_{10}\ ]$$

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octave]
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2. Compute the mean of the measurements $\bar{\boldsymbol{y}}$, and construct the centered data matrix

$$\boldsymbol{Y}_0 = [\ \boldsymbol{y}_1 - \bar{\boldsymbol{y}}\ \ \boldsymbol{y}_2 - \bar{\boldsymbol{y}}\ \ \dots\ \ \boldsymbol{y}_{10} - \bar{\boldsymbol{y}}\ ]$$

3. Compute the first 3 singular vectors of $\boldsymbol{Y}_0$ using the `svds` Octave function.

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octave]
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4. The largest 10 singular values are found through the instruction `sigma=svds(Y0,10,'L')`. Display these values and comment on your observations.

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octave]
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5. Plot the first 3 singular vectors. What features of the data $\boldsymbol{Y}$ is revealed by the dominant singular vectors?

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octave]
```

Figure 2. Place your plot of the first three singular vectors here using menu item Insert->Image->Insert image …