1. MATH347 HOMEWORK 1

Topic:	Linear combinations, measuring vectors
Post date:	May 17, 2024
Due date:	May 21, 2024

1.1. Background

This homework is meant as a tutorial on matrix computations, in particular:

- vector addition and scaling;
- vector norms;
- inner products;
- matrix-vector products;
- linear mapping matrices;
- linear mapping composition matrix-matrix products.

1.2. Theoretical questions

1.2.1. Vector addition and scaling

Problem. Consider n gray-scale images of size $p \times q$ pixels, where the brightness value of pixel at position (i, j), $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant q$ is $0 \leqslant b_{ij}^{(l)} < 2^{R+1}$, $1 \leqslant l \leqslant n$, $R \in \mathbb{N}_+$.

a) Represent the images as column vectors $\boldsymbol{a}_k$ of a matrix $\boldsymbol{A} = [\ \boldsymbol{a}_1\ \cdots\ \boldsymbol{a}_n\ ] = [a_{kl}]$. What is the matrix size?

b) Construct a composite image $\boldsymbol{b}$ obtained by taking fractions $0 \leqslant x_l \leqslant 1$ of image l, $1 \leqslant l \leqslant n$, $x_l \in [0, 1]$.

Answer.

1.2.2. Norms in E_m

Problem. Verify whether for $a_1, a_2 > 0$, $f: \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$f(\boldsymbol{x}) = a_1 x_1^2 + a_2 x_2^2,$$

is a norm in E_2.

Answer.

1.2.3. Norms in E_m

Problem. Verify whether for $a_1, a_2 > 0$, $f: \mathbb{R}^2 \to \mathbb{R}_+$ defined by

$$f(\boldsymbol{x}) = a_1 x_1^2 - a_2 x_2^2$$

is a norm in E_2.

Answer.

1.2.4. Inner product in E_m

Problem. Verify whether for $a_1 > 0, a_2 > 0$, $s\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$s(\boldsymbol{x}, \boldsymbol{y}) = a_1 x_1 y_1 + a_2 x_2 y_2,$$

is a scalar product in E_2.

Answer.

1.2.5. Inner product in E_m

Problem. Verify whether for $a_1, a_2 > 0$, $s\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$s(\boldsymbol{x}, \boldsymbol{y}) = a_1 x_1 y_1 - a_2 x_2 y_2,$$

is a scalar product in E_2.

Answer.

1.2.6. Matrix-matrix products

Problem. Verify that the "row-over-columns" matrix multiplication rule is equivalent to columns of matrix-vector multiplications, i.e., for $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{B} \in \mathbb{R}^{n\times p}$, $\boldsymbol{C} \in \mathbb{R}^{m\times p}$

$$\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{\text{mn}} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{bmatrix} = \boldsymbol{A}[\ \boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \dots\ \boldsymbol{b}_p\ ] = [\ \boldsymbol{A}\boldsymbol{b}_1\ \ \boldsymbol{A}\boldsymbol{b}_2\ \dots\ \boldsymbol{A}\boldsymbol{b}_p\ ]$$

Answer.

1.2.7. Linear mapping representation

Problem. Determine the matrix $\boldsymbol{C}$ of the linear mapping $\boldsymbol{h}: \mathbb{R}^2 \to \mathbb{R}^2$ representing rotation of a vector by angle θ followed by reflection across the x_2-axis. Let $\boldsymbol{A}, \boldsymbol{B}$ denote the matrices of the mappings $\boldsymbol{f}, \boldsymbol{g}$ corresponding to: (1) reflection by angle θ; (2) reflection across the x_2-axis. Verify that $\boldsymbol{h} = \boldsymbol{g} \circ \boldsymbol{f}$ corresponds to $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$. Does $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$?

Answer.

1.2.8. Linear mapping composition

Problem. Construct the rotation matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ for rotation by angles $\theta, \varphi, \theta + \varphi$. Verify $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$, and the trigonometric identities for sine and cosine of sums of angles. Does $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$? Compare your answer to previous case.

Answer.

1.2.9. Block matrix operations

Problem. Identify repeated blocks in the matrices $\boldsymbol{A},\boldsymbol{B}$ and carry out the mutiplication in Julia: (1) component-wise; (2) by blocks. Compare the two results.

$$\boldsymbol{C}=\boldsymbol{A}\boldsymbol{B}^T=\begin{bmatrix} 2 & 1 & 0 & 3 \\ 3 & 2 & 3 & 0 \\ 0 & 3 & 2 & 1 \\ 3 & 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 \end{bmatrix}^T$$

Answer.

1.2.10. Block matrix operations

Problem. Carry out the following block matrix multiplication

$$\boldsymbol{A}=\boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T=[\ \boldsymbol{u}_1\ \ \boldsymbol{u}_2\ \ \dots\ \ \boldsymbol{u}_m\ ]\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_n \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}[\ \boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \dots\ \ \boldsymbol{v}_n\ ]^T.$$

Answer.

1.3. Linear functionals and mappings in analysis of EEG data

Electroencephalograms (EEGs) are recordings of the electric potential on cranium skin. Research on brain activity uses EEGs to determine specific activity patterns in the brain. For example, epileptic seizures have a distinctive EEG signature. EEG data can be loaded from the course data directory. As is typically the case in data analysis, an additional package is required to read in the particular data format; in this case the data is given in Matlab `.mat` format. This need only be done once.

```
∴ import Pkg; Pkg.add("MAT");
```

```
    Updating registry at `~/.julia/registries/General.toml`
   Resolving package versions...
  No Changes to `~/.julia/environments/v1.10/Project.toml`
  No Changes to `~/.julia/environments/v1.10/Manifest.toml`
```

Load EEG data using the Julia package to read Matlab files. There are $n=32$ electrode recordings at $m=30504$ moments of time. A common first step in working with data is centering and scaling values to a standard range.

```
∴ using MAT
```

```
∴ DataFileName = homedir()*"/courses/MATH347DS/data/eeg/eeg.mat";
```

```
∴ DataFile = matopen(DataFileName,"r");
```

```
∴ dict = read(DataFile,"EEG");
```

```
∴ data = dict["data"]';
```

```
∴ m,n=size(data)
```

$$\begin{bmatrix} 30504 \\ 32 \end{bmatrix} \tag{1}$$

```
∴ mx=findmax(data[:,4:28])[1]; mn=findmin(data[:,4:28])[1];
∴ d = 2*(data .- mn)/(mx-mn) .- 1;
∴ clf();
∴ for j=1:n
    plot(d[:,j].+(j-1)*2)
  end
∴ xlabel("time"); ylabel("voltage"); title("EEG␣data");
∴ cd(homedir()*"/courses/MATH347DS/homework/hw01");
∴ savefig("H01Fig01.eps");
∴
```

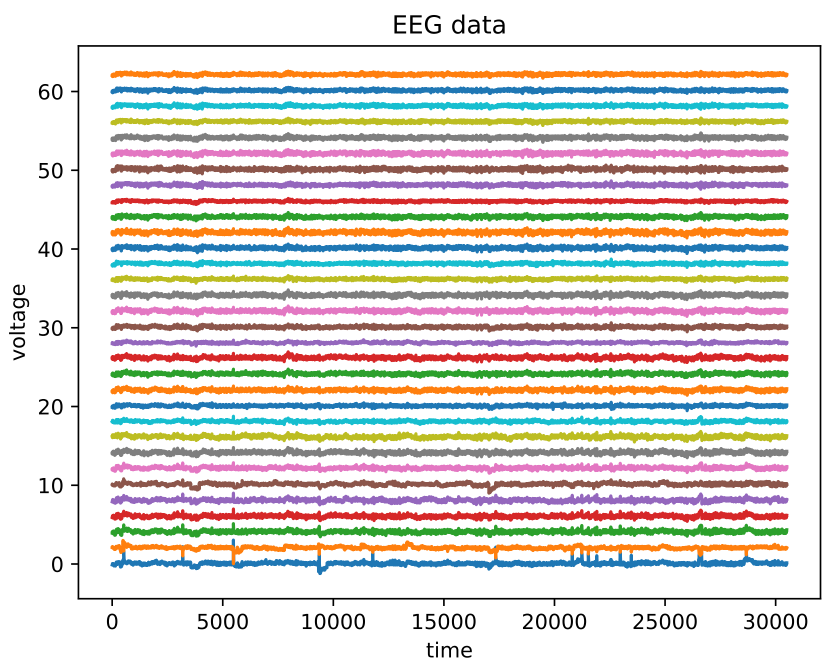

Figure 1. Electrode voltage time history in EEG

```
∴ clf();
∴ for j=1:4
    plot(d[1000:1500,j].+(j-1)*2)
  end
∴ xlabel("time");    ylabel("voltage");    title("EEG␣data␣portion");
  grid("on");
∴ savefig("H01Fig02.eps");
∴
```

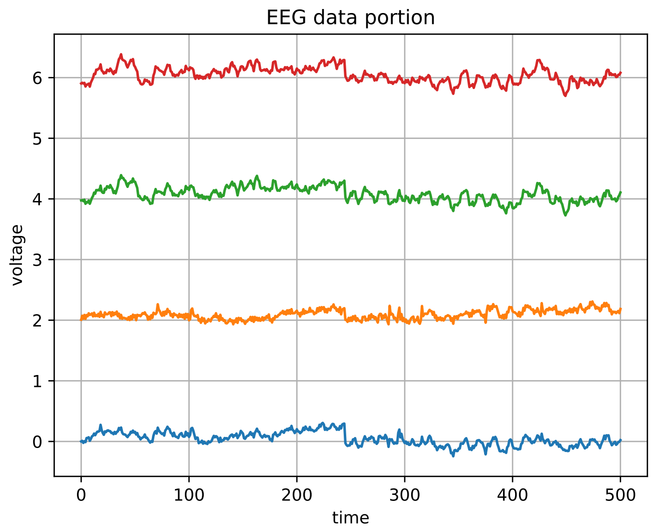

Figure 2. Detail of recordings from first four electrodes.

1.3.1. Functional relationships in the data (functions)

Over some time subinterval, can the data from a sensor be expressed as a function of data from another sensor?

1.3.2. Data magnitude (norm)

Partition sensor data into time subintervals. Over each subinterval, evaluate the p-norm of centered data, for $p = 1, 2,$ 3, $p \to \infty$. Plot the p-norms over the entire time history.

1.3.3. Orthogonality of sensor data (inner product)

Over some time interval, determine the angle between sensor data. Is sensor data orthogonal? Plot the angle between sensor data over the entire time history.

1.3.4. Linear relationships in the data (linear mapping)

Can data from a sensor be expressed as a linear combination of other sensors?