

1. MATH347 HOMEWORK 3

Topic: SVD identification of artwork features

Post date: May 28, 2024

Due date: May 29, 2024

1.1. Background

This homework investigates consequences of the fundamental theorem of algebra and application of the singular value decomposition.

Up to now, coding in Julia has been considered in a command prompt environment (Console, Terminal), and as sessions embedded in TeXmacs. This assignment introduces the native Julia notebook environment, which is called Pluto and is hosted in a browser window. Choice of environment is influenced by code documentation requirements (Table 1). A number of other environments exist (JupyterLab, VSCode, Emacs, Juno), and there is no universal “best” environment. Rather a choice is made according to purpose of the code and personal preference. For this assignment work can be carried out either in TeXmacs or Pluto.

| Environment | Host | Quick calculations, Adding packages | Many figures, Moderate code documentation | Few, key figures. Publication quality documentation |
|------------------|-------------------------|-------------------------------------|---|---|
| Command line IDE | Terminal, VSCode, Emacs | Excellent | Inconvenient | Inconvenient |
| TeXmacs session | Texmacs | Very good | Inconvenient | Difficult |
| Pluto notebook | Browser | Inconvenient | Good | Excellent |
| | | | Excellent | Difficult |

Table 1. Comparison of different types of Julia environments. Similar considerations for Python.

To use the Pluto notebook format, start a console and type:

```
julia
using Pluto
Pluto.run(notebook="hw03.jl")
```

1.2. Theoretical questions

Consider a linear mapping $f:U \rightarrow V$, from vector space $\mathcal{U}=(U, \mathbb{R}, +, \cdot)$ with basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$, to $\mathcal{V}=(V, \mathbb{R}, +, \cdot)$, with basis $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$.

1. Is $\{f(\mathbf{u}_1), \dots, f(\mathbf{u}_n)\}$ a basis for \mathcal{V} ? (1 point)
2. If $\text{nullity}(f)=0$ must $m=n$? (1 point)
3. If $m=n$ and $\mathbf{u}_i=\mathbf{v}_i$ for $i=1, \dots, m$, what is the matrix A representing f ? (2 points)
4. Determine the singular value decomposition and pseudo-inverse of a matrix $A \in \mathbb{R}^{1 \times n}$ (i.e., a row vector). (2 points)

1.3. Ordered bases for the fundamental spaces and painting motifs

1.3.1. Introduction

The fundamental theorem of linear algebra partitions the domain and codomain of a linear mapping. The singular value decomposition provides orthogonal bases for each of the subspaces arising in the partition. The bases are ordered according to the amplification behavior of the linear mapping, expressed through the norm of successive restrictions of the mapping. This approach is closely aligned with typical problems in data science, and can be used in a variety of scenarios. In this homework linear algebra methods will first be used in a field far removed from the physical sciences: extracting the quirks of painter style from the overall composition of a painting, and applying one artist's style to another artist's composition. This is often-encountered data science problem: distinguishing between small and large scale features of data.

MATH547 Homework

```

∴ mkpath(homedir()*"/courses/MATH347DS/homework/hw03");
∴ cd(homedir()*"/courses/MATH347DS/homework/hw03");
∴ using Images

```

Load an image, transform it to a matrix, note the image size, and compute the SVD of the matrix.

```

∴ imFK96=Gray.(load("./paintings/Frida_Kahlo_96.jpg"));
∴ AFK96=Real.(imFK96);
∴ mx,my=size(AFK96)

```

$$\begin{bmatrix} 484 \\ 430 \end{bmatrix} \quad (1)$$

```

∴ F=svd(AFK96);
∴ norm(AFK96-F.U*Diagonal(F.S)*F.Vt)

```

0.0002369014

```
∴
```

Load another image, select a portion of the same size as previous image and compute its SVD.

```

∴ imVvG95=Gray.(load("./paintings/Vincent_van_Gogh_95.jpg"));
∴ size(imVvG95)

```

$$\begin{bmatrix} 614 \\ 535 \end{bmatrix} \quad (2)$$

```

∴ AVvG95=Real.(imVvG95)[1:mx,1:my];
∴ G=svd(AVvG95);
∴

```

Note that the singular values of the two images decay rapidly

```

∴ clf(); plot(1:my,F.S,".b",1:my,G.S,".r"); xlabel("i"); ylabel(L"\sigma_i");
∴ savefig("hw03f01.eps");
∴

```

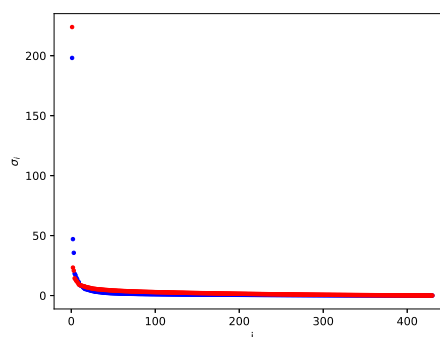


Figure 1. Singular values σ_i of the two images.

Generate a new image by combining the large-scale features of the first image and the small-scale features of the second. Choose the new image to reflect 70% ($x_1=0.3$ of the composition of the first image and 30% ($x_2=0.7$) of the small-scale features of the second. Choose large scale features of the first image to include rank-1 updates from 1 to k_A . Choose small scale features of the second image to include rank-1 updates from k_B to l_B .

```
∴ x1=0.3; x2=0.7; kA=40; kB=80; lB=120;
```

Form the large-scale features from first image. Note that V_t contains the transposed V matrix.

```
∴ A1 = F.U[:,1:kA]*Diagonal(F.S[1:kA])*F.Vt[1:kA,:];
```

Form the small-scale features from second image

```
∴ B1 = G.U[:,kB:lB]*Diagonal(G.S[kB:lB])*G.Vt[kB:lB,:];
```

Combine the two to obtain a new image, display the images and save them.

```
∴ Anew=x1*A1+x2*B1;
```

```
∴ figure(1); clf(); imshow(A1,cmap="gray"); savefig("hw03A1.png");
```

```
∴ figure(2); clf(); imshow(B1,cmap="gray"); savefig("hw03B1.png");
```

```
∴ figure(3); clf(); imshow(Anew,cmap="gray"); savefig("hw03Anew.png");
```

```
∴
```

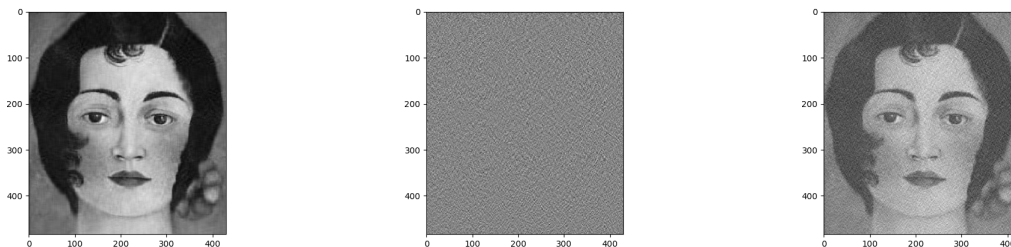


Figure 2. A new painting (right) reflecting the composition of the Frida Kahlo self-portrait (left) with the brush stroke extracted from the Vincent van Gogh self-portrait (middle).

1.3.2. Tasks

Generate 4 different new paintings (3 points each) experimenting with various images as presented above. Comment each result.