DATA STABILITY

$$A = U\Sigma V^T \quad A = QR \quad A = LU$$

1. The eigenvalue problem

- Consider square matrix $A \in \mathbb{R}^{m\times m}$. The eigenvalue problem asks for vectors $x \in \mathbb{C}^m$, $x \neq 0$, scalars $\lambda \in \mathbb{C}$ such that

$$Ax = \lambda x \tag{1}$$

- Eigenvectors are those special vectors whose direction is not modified by the matrix A
- Rewrite (1): $(A - \lambda I)x = 0$, and deduce that $A - \lambda I$ must be singular in order to have non-trivial solutions $\det(A - \lambda I) = 0$
- Consider the determinant

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} - \lambda \end{vmatrix}$$

- From determinant definition ``sum of all products choosing an element from row/column''

$$\det(A - \lambda I) = (-1)^m \lambda^m + c_1 \lambda^{m-1} + \ldots + c_{m-1}\lambda + c_m = p_A(\lambda)$$

is the characteristic polynomial associated with the matrix A, and is of degree m

- $A \in \mathbb{R}^{m\times m}$ has characteristic polynomial $p_A(\lambda)$ of degree m, which has m roots (Fundamental theorem of algebra)
- Example

```
octave] theta=pi/3.; A=[cos(theta) -sin(theta); sin(theta) cos(theta)]
```

```
A =

   0.50000  -0.86603
   0.86603   0.50000
```

```
octave] eig(A)
```

```
ans =

   0.50000 + 0.86603i
   0.50000 - 0.86603i
```

```
octave] [R,lambda]=eig(A);
```

```
octave] disp(R);
```

```
   0.70711 + 0.00000i   0.70711 - 0.00000i
   0.00000 - 0.70711i   0.00000 + 0.70711i
```

```
octave] disp(lambda)
```

```
Diagonal Matrix

   0.50000 + 0.86603i                    0
                    0   0.50000 - 0.86603i
```

```
octave] A=[-2 1 0 0 0 0; 1 -2 1 0 0 0; 0 1 -2 1 0 0; 0 0 1 -2 1 0; 0 0 0 1 -2 1; 0 0
        0 0 1 -2];
```

```
octave] disp(A)
```

```
  -2   1   0   0   0   0
   1  -2   1   0   0   0
   0   1  -2   1   0   0
   0   0   1  -2   1   0
   0   0   0   1  -2   1
   0   0   0   0   1  -2
```

```
octave] lambda=eig(A);
```

```
octave] disp(lambda);
```

```
  -3.80194
  -3.24698
  -2.44504
  -1.55496
  -0.75302
  -0.19806
```

```
octave]
```

- For $A \in \mathbb{R}^{m \times m}$, the eigenvalue problem 5 ($x \neq 0$) can be written in matrix form as

$$AX = X\Lambda, X = (x_1 \ldots x_m) \text{ eigenvector}, \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m) \text{ eigenvalue matrices}$$

- If the column vectors of X are linearly independent, then X is invertible and A can be reduced to diagonal form

$$A = X\Lambda X^{-1}, A = U\Sigma V^T$$

- Diagonal forms are useful in solving linear ODE systems

$$y' = \text{Ay} \Leftrightarrow (X^{-1}y) = \Lambda\ (X^{-1}y)$$

- Also useful in repeatedly applying A

$$u_k = A^k u_0 = AA\ldots Au_0 = (X\Lambda X^{-1})(X\Lambda X^{-1})\ldots(X\Lambda X^{-1})u_0 = X\Lambda^k X^{-1}u_0$$

- When can a matrix be reduced to diagonal form? When eigenvectors are linearly independent such that the inverse of X exists

- Matrices with distinct eigenvalues are diagonalizable. Consider $A \in \mathbb{R}^{m \times m}$ with eigenvalues $\lambda_j \neq \lambda_k$ for $j \neq k$, $j, k \in \{1, \ldots, m\}$
  Proof. By contradiction. Take any two eigenvalues $\lambda_j \neq \lambda_k$ and assume that x_j would depend linearly on x_k, $x_k = cx_j$ for some $c \neq 0$. Then

$$Ax_1 = \lambda_1 x_1 \ \Rightarrow \ Ax_1 = \lambda_1 x_1$$
$$Ax_2 = \lambda_2 x_2 \ \Rightarrow \ Acx_1 = \lambda_2 cx_1$$

  and subtracting would give $0 = (\lambda_1 - \lambda_2)x_1$. Since x_1 is an eigenvector, hence $x_1 \neq 0$ we obtain a contradiction $\lambda_1 = \lambda_2$.

- The characteristic polynomial might have repeated roots. Establishing diagonalizability in that case requires additional concepts

DEFINITION 1. *The algebraic multiplicity of an eigenvalue λ is the number of times it appears as a repeated root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$*

Example. $p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ has two single roots $\lambda_1 = 0$, $\lambda_2 = 1$ and a repeated root $\lambda_{3,4} = 2$. The eigenvalue $\lambda = 2$ has an algebraic multiplicity of 2

DEFINITION 2. *The geometric multiplicity of an eigenvalue λ is the dimension of the null space of $A - \lambda I$*

DEFINITION 3. *An eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity is said to be defective*

PROPOSITION 4. *A matrix is diagonalizable is the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity of that eigenvalue.*

- Finding eigenvalues as roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ is suitable for small matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$.
  - analytical root-finding formulas are available only for $m \leqslant 4$
  - small errors in characteristic polynomial coefficients can lead to large errors in roots
- Octave/Matlab procedures to find characteristic polynomial
  - `poly(A)` function returns the coefficients
  - `roots(p)` function computes roots of the polynomial

```
octave] A=[5 -4 2; 5 -4 1; -2 2 -3]; disp(A);
```

```
 5  -4   2
 5  -4   1
-2   2  -3
```

```
octave] p=poly(A); disp(p);
```

```
 1.00000   2.00000  -1.00000  -2.00000
```

```
octave] r=roots(p); disp(r');
```

```
 1.0000  -2.0000  -1.0000
```

```
octave]
```

- Find eigenvectors as non-trivial solutions of system $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = 0$

$$\lambda_1 = 1 \Rightarrow \boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 1 \\ -2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 5 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

  Note convenient choice of row operations to reduce amount of arithmetic, and use of knowledge that $\boldsymbol{A} - \lambda_1 \boldsymbol{I}$ is singular to deduce that last row must be null

- In traditional form the above row-echelon reduced system corresponds to

$$\begin{cases} -2x_1 + 2x_2 - 4x_3 &= 0 \\ 0x_1 + 0x_2 - 6x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{cases} \Rightarrow \boldsymbol{x} = \alpha \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \|\boldsymbol{x}\| = 1 \Rightarrow \alpha = 1/\sqrt{2}$$

- In Octave/Matlab the computations are carried out by the null function

```
octave] null(A+5*eye(3))'
```

```
ans = [](0x3)
```

```
octave]
```

- The eigenvalues of $\boldsymbol{I} \in \mathbb{R}^{3 \times 3}$ are $\lambda_{1,2,3} = 1$, but small errors in numerical computation can give roots of the characteristic polynomial with imaginary parts

```
octave> lambda=roots(poly(eye(3))); disp(lambda')
```

```
   1.00001 - 0.00001i   1.00001 + 0.00001i   0.99999 - 0.00000i
```

```
octave>
```

- In the following example notice that if we slightly perturb $\boldsymbol{A}$ (by a quantity less than 0.0005=0.05%), the eigenvalues get perturb by a larger amount, e.g. 0.13%.

```
octave] A=[-2 1 -1; 5 -3 6; 5 -1 4]; disp([eig(A) eig(A+0.001*(rand(3,3)-0.5))])
```

```
   3.0000 + 0.0000i   3.0005 + 0.0000i
  -2.0000 + 0.0000i  -2.0000 + 0.0161i
  -2.0000 + 0.0000i  -2.0000 - 0.0161i
```

```
octave]
```

- Extracting eigenvalues and eigenvectors is a commonly encountered operation, and specialized functions exist to carry this out, including the `eig` function

```
octave>  [X,L]=eig(A); disp([L X]);
```

```
  -2.00000   0.00000   0.00000  -0.57735  -0.00000   0.57735
   0.00000   3.00000   0.00000   0.57735   0.70711  -0.57735
   0.00000   0.00000  -2.00000   0.57735   0.70711  -0.57735
```

```
octave> disp(null(A-3*eye(3)))
```

```
   0.00000
   0.70711
   0.70711
```

```
octave> disp(null(A+2*eye(3)))
```

```
   0.57735
  -0.57735
  -0.57735
```

```
octave>
```

- Recall definitions of eigenvalue algebraic $\boldsymbol{m}_\lambda$ and geometric multiplicities $\boldsymbol{n}_\lambda$.

DEFINITION. *A matrix which has $n_\lambda < m_\lambda$ for any of its eigenvalues is said to be defective.*

```
octave> A=[-2 1 -1; 5 -3 6; 5 -1 4]; [X,L]=eig(A); disp(L);
```

```
Diagonal Matrix

  -2.0000        0        0
        0   3.0000        0
        0        0  -2.0000
```

```
octave> disp(X);
```

```
  -5.7735e-01  -1.9153e-17   5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
   5.7735e-01   7.0711e-01  -5.7735e-01
```

```
octave> disp(null(A+2*eye(3)));
```

```
   0.57735
  -0.57735
  -0.57735
```

```
octave> disp(rank(X))
  2
octave>
```

2. Computation of the SVD

- The SVD is determined by eigendecomposition of A^TA, and AA^T
  - $A^TA = (U\Sigma V^T)^T\,(U\Sigma V^T) = V\,(\Sigma^T\Sigma\,)\,V^T$, an eigendecomposition of A^TA. The columns of V are eigenvectors of A^TA and called right singular vectors of A

$$B = A^TA = V\Sigma^T\Sigma V^T = V\Lambda V^T$$

  - $AA^T = (U\Sigma V^T)(U\Sigma^T V^T)^T = U\,(\Sigma\Sigma^T)\,U^T$, an eigendecomposition of A^TA. The columns of U are eigenvectors of AA^T and called left singular vectors of A
  - The matrix Σ has form

$$\Sigma = \begin{pmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & & & & 0 & \\ & & & & & \ddots \end{pmatrix} \in \mathbb{R}_+^{m\times n}$$

    and σ_i are the singular values of A.
- The singular value decomposition (SVD) furnishes complete information about A
  - $\text{rank}(A) = r$ (the number of non-zero singular values)
  - U, V are orthogonal basis for the domain and codomain of A