Linear combinations - the basic idea

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Linear algebra: subfield of algebra (object of study: operations)
linear combinations involving scalars, vectors
(fundamental objects)

Vector space

$\vartheta = (V, S, +, \cdot)$

ϑ = vector space
(algebraic structure)

has 4 parts

V set of vectors ($\mathbb{R}^m$)

S set of scalars ($\mathbb{R}$)

$+$ for vectors in $\mathbb{R}^m$
(component wise addition)

$a, b \in \mathbb{R}^m$

$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$c = a + b = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_m+b_m \end{bmatrix}$

$x \in S = \mathbb{R}$

$x a = \begin{bmatrix} xa_1 \\ xa_2 \\ \vdots \\ xa_m \end{bmatrix}$

Example 1

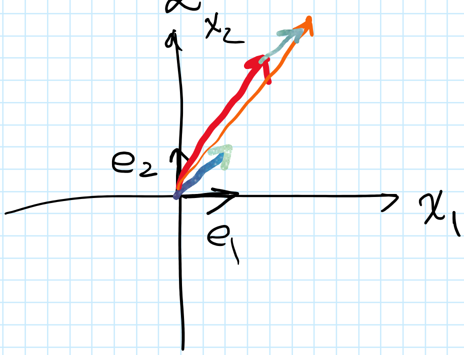
$$x\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \Rightarrow x = 1$$

By definition $a, b \in \mathbb{R}^m$ are equal ($a = b$)
$\rightarrow$ componentwise equality

$\underbrace{a_1 = b_1, \ldots, a_m = b_m}$, e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\mathbb{R}^m$ vector equality $\Longleftrightarrow$ m scalar equalities

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \in \mathbb{R}^2$

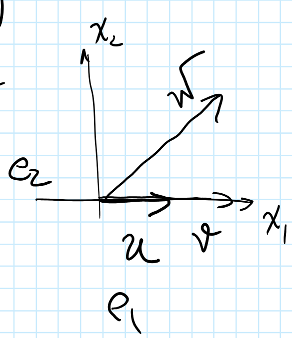


Example 2:

$$\nexists x \text{ s.t. } \underbrace{x\begin{bmatrix} 1 \\ 1 \end{bmatrix}} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Question 1) What can be reached by linear combinations?

Example 3



$u \in \mathbb{R}^2$ $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v \in \mathbb{R}^2$, $w \in \mathbb{R}^2$

$v = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $v = x u$ $x \in \mathbb{R}$

$x = 2$

$w = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$; $w = y u \Rightarrow$ $2 = y \cdot 1$

$2 = y \cdot 0 \Rightarrow \nexists y$

$e_1 = u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e_1 = u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Can w be reached by linear combination ~~of~~ e_1 e_2

$$w = \underbrace{\mu e_1 + \nu e_2} \qquad \mu, \nu \in \mathbb{R} \qquad \begin{aligned} \mu &= 2 = w_1 \\ \nu &= 2 = w_2 \end{aligned}$$

$I = [e_1 \ e_2]$ $\qquad w = I \cdot \begin{bmatrix} \mu \\ \nu \end{bmatrix}$ $\qquad I \cdot w = I \cdot \begin{bmatrix} \mu \\ \nu \end{bmatrix} \Rightarrow \begin{bmatrix} \mu \\ \nu \end{bmatrix} = w.$

$w = I \cdot w$

Question 2: (see computer example)

If we can't exactly reach a specific vector by linear combination, how close can we get?