Functions, linear mappings, norms, scalar products

Thursday, May 18, 2023 1:08 PM

Function (D, C, f) — domain, codomain, name of function

$f: \underline{D} \to \underline{C}$

Ex 1: $f: \mathbb{R} \to \mathbb{R}$, $f(t) = \sin(t)$

Ex 2: $g: \mathbb{R}_+ \to \mathbb{R}$, $g(t) = \sin(t)$

$f \overset{?}{\cancel{=}} g$

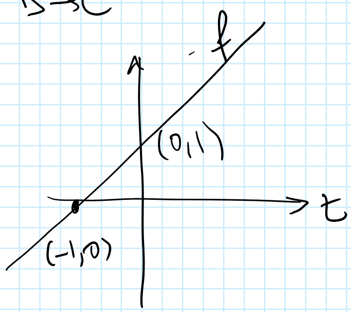
linear functions

f is linear if $\boxed{f}(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$f: D \to C$

$\forall\, x, y \in D$ α, β are scalars

Ex.1:



$$f(t) = t + 1$$

$$f(\alpha x + \beta y) = \alpha x + \beta y + 1$$

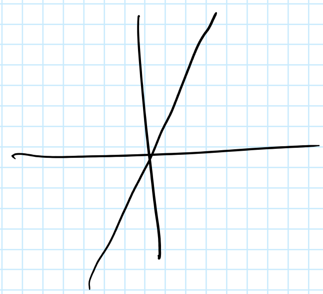
$$f(\alpha x) = \alpha x + 1 \qquad f(\beta y) = \beta y$$

$$f(x) = x + 1 \qquad f(y) = y + 1$$

$$f(\alpha x + \beta y) \overset{?}{=} \alpha f(x) + \beta f(y)$$

$$\alpha x + \beta y + 1 \overset{?}{=} \alpha(x+1) + \beta(y+1) = \alpha x + \beta y + \alpha + \beta$$

Ex: $g(t) = 3t$



$$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$$

$$g(\alpha x + \beta y) = 3(\alpha x + \beta y) \qquad g(x) = 3x \quad g(y) = 3y$$

$$3(\alpha x + \beta y) \overset{?}{=} \alpha(3x) + \beta(3y) \quad \checkmark$$

$3(\alpha x + \beta y) \stackrel{?}{=} \alpha(3x) + \beta(3y)$ ✓

Mapping : function between vector spaces

$(\mathbb{R}^{n}, \mathbb{R}^{m}, f)$ $\quad f: \mathbb{R}^n \to \mathbb{R}^m$

Linear Mapping : $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

Functional is a function from a vector space to a scalar

$f: \mathbb{R}^m \to \mathbb{R}$

Ex: 1 Norms: 1.a. Euclidean (2-norm)

$\| \ \|_2 : \mathbb{R}^m \to \mathbb{R}_+$

$x \in \mathbb{R}^m \quad \|x\|_2 = (x_1^2 + x_2^2 + \dots + x_m^2)^{1/2}$

1.b. 1-norm

$\| \ \|_1 : \mathbb{R}^m \to \mathbb{R}_+$

$x \in \mathbb{R}^m \quad \|x\|_1 = (|x_1| + |x_2| + \dots + |x_m|)$

1.c. $p \in \mathbb{N}$ $\quad$ p-norm

$x \in \mathbb{R}^m \quad \|x\|_p = (|x_1|^p + |x_2|^p + \dots + |x_m|^p)^{1/p}$

1.d. p gets really large $\quad p = \infty$

$x \in \mathbb{R}^m \quad \|x\|_\infty = \max_{1 \le j \le m} |x_j|$

Ex. 2 Scalar products

$$(\ , \ ) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

2.a $\quad u, v \in \mathbb{R}^m \qquad (u, v) = \sum_{i=1}^{m} u_i v_i$

2.b. $\quad u, v \in \mathbb{R}^m \qquad (u, v) = u^T v$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}; \quad u^T = [u_1 \ \ u_2 \ \dots \ u_m]$$

$$u^T v = [u_1 \ \ u_2 \ \dots \ u_m] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_m v_m$$

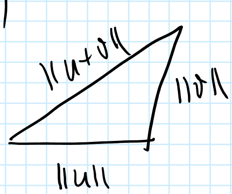
Norm properties

$\| \ \| : \mathbb{R}^m \to \mathbb{R}_+ \quad u, v \in \mathbb{R}^m, a \in \mathbb{R}$

1) $\|a v\| = |a| \, \|v\|$

2) $\|v\| = 0 \Rightarrow v = 0$

3) $\|u + v\| \leq \|u\| + \|v\|$



Scalar (Inner) product properties

$(\ , \ ) : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \quad u, v \in \mathbb{R}^m$

1) $(u, v) = (v, u)$

2) $(u, \alpha v + \beta w) = \alpha (u, v) + \beta (u, w)$

3) $(u, u) \geq 0$

$(u, u) = 0 \Rightarrow u = \mathbb{0}$