Vector subspaces, matrix fundamental spaces

Friday, May 19, 2023 11:29 AM

Vector space $\vartheta = (V, S, +, \cdot)$

$E_m = (\mathbb{R}^m, \mathbb{R}, +, \cdot)$ (Euclidean)

$\forall a, b, c \in V \quad \forall \alpha, \beta \in S$

$a + b \in V$

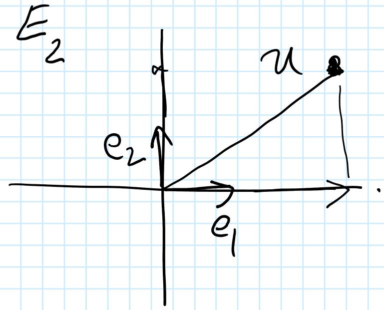
$(a+b)+c = a + (b+c)$

$a + 0 = a$

$a + (-a) = 0$

....

Ex. E_2



$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u = \underbrace{I \cdot u}$$

matrix vector product

$\Rightarrow$ linear combination

$$u = u_1 e_1 + u_2 e_2$$

$\mathcal{U} = (U, S, +, \cdot)$ vector subspace is closed under linear combination

$$E_2 = (\mathbb{R}^2, \mathbb{R}, +, \cdot)$$

$$\mathbb{R} \subset \mathbb{R}^2$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$E_1 = (\underset{\smile}{\mathbb{R}}, \mathbb{R}, +, \cdot)$$

$\hookrightarrow$ along horizontal

From set theory (primer)

def $A \subset B$ means that $\forall a \in A$, $a \in B$

$\mathbb{R} \overset{?}{\subset} \mathbb{R}^2 \qquad \forall x \in \mathbb{R} \qquad \cancel{x = \begin{pmatrix} x \\ 0 \end{pmatrix}}$

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def Vector span = "set of vectors reachable by linear combination of some finite set of vectors."

$S = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_n\} \qquad \vartheta_1 \in V, \ \vartheta_2 \in V, \ldots, \vartheta_n \in V$

$$\mathrm{Span}(S) = \{ \underline{u} \mid \underbrace{\exists a_1, a_2, \dots, a_n \in S}\ \text{s.t.}\ u = a_1 v_1 + a_2 v_2 + \dots + a_n v_n \}$$

$$\mathbf{V} = [v_1\ v_2\ \dots\ v_n] \ ; \ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} ; \quad u = \mathbf{V}\mathbf{a}$$

<u>def</u> Column space of a matrix $A \in \mathbb{R}^{m \times n}$ (n column vectors, m components in each)

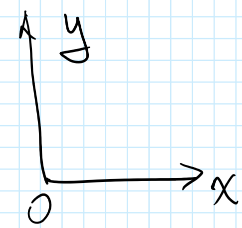
$$C(A) = \{ \underline{y} \mid \exists x \in \mathbb{R}^n \text{ s.t. } y = A\underline{x} \in \mathbb{R}^m \}$$

$C(A)$ is a set of vectors

$$C(A) \subseteq \mathbb{R}^m$$

- $C\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R} \right\}$ $\qquad \begin{bmatrix} x \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$C\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R} \right\}$



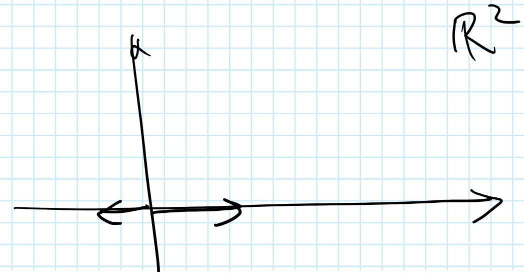
$C\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = C(e_1)$ subset of $\mathbb{R}^2$? Yes!

$C\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = C(e_2)$ subset of $\mathbb{R}^2$? Yes!

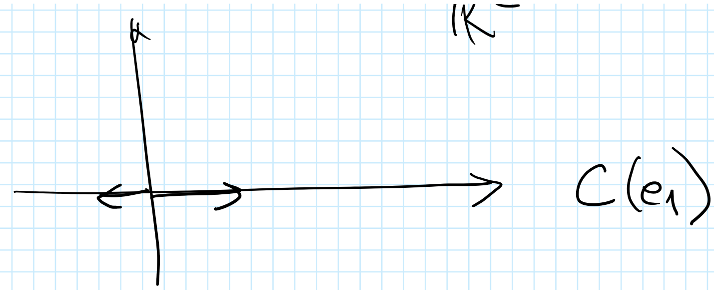
$E_2 = (\mathbb{R}^2, \mathbb{R}, +, \cdot)$ is a vector space

$C(e_1)$ is it a vector subspace of $\mathbb{R}^2$ Yes!

$\mathbb{R}^2$



$C(e_1)$



$C(e_1) < \mathbb{R}^2$ $\hookrightarrow$ vector subspace notation

$C(e_2) \overset{?}{<} \mathbb{R}^2$ Yes

Sum of vector spaces

$\mathcal{U} = (U, S, +, \cdot)$ vector space

$\mathcal{V} = (V, S, +, \cdot)$ — " —

$\mathcal{W} = \mathcal{U} + \mathcal{V} = (W, S, +, \cdot)$

$W = \{ w \mid \exists\, u \in U \text{ and } \exists\, v \in V \text{ s.t. } w = u + v \}$

Ex: $\mathbb{R}^2 = C(e_1) + C(e_2)$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ y \end{bmatrix}$

$C(A) = \{ y \in \mathbb{R}^m \mid \exists\, x \in \mathbb{R}^n \text{ s.t. } y = Ax \}$ $\quad A \in \mathbb{R}^{m \times n}$

$\hookrightarrow$ "set of reachable vectors"

Seek a definition for the set of unreachable vectors

Ex: $\mathbb{R}^3$ $\quad e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $\quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Ex: $\mathbb{R}^3$ $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$A = [e_1 \; e_2] \in \mathbb{R}^{3\times 2} \qquad \begin{matrix} m = 3 \\ n = 2 \end{matrix}$$

$$C(A) \subset \mathbb{R}^3$$

There is a set of vectors within $\mathbb{R}^3$ not reachable by linear combination of columns of A.

$u, v \in \mathbb{R}^m$ are orthogonal if $u^T v = 0$

$(u \cdot v = \|u\| \|v\| \cdot \cos\theta)$

$\cdot \; e_1^T e_3 = e_3^T e_1 = 0 = [0 \;\; 0 \;\; 1] \begin{bmatrix}1\\0\\0\end{bmatrix} = 0\cdot 1 + 0 \cdot 0 + 1 \cdot 0 = 0$

$\cdot \; e_2^T e_3 = e_3^T e_2 = 0 = [0 \; 0 \; 1] \begin{bmatrix}0\\1\\0\end{bmatrix} = 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0$

$$A = [e_1 \; e_2]; \qquad A^T = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^T e_3 = \begin{bmatrix} e_1^T \\ e_2^T \end{bmatrix} (e_3) = \begin{bmatrix} e_1^T e_3 \\ e_2^T e_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Def Null space of A^T is

$N(A^T) = \{ \ldots$ "the unreachable

Def Null space of A^T is

$$N(A^T) = \{ y \mid A^T y = 0 \}$$ "The unreadable set"

Preview of Fundamental Theorem of Linear Algebra

Linear Mapping $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$ $\quad \underline{\underline{y}} = f(x) = \underline{\underline{A}}\,\underline{\underline{x}}$

$$A \in \mathbb{R}^{m \times n}$$

$C(A)$ column space of A

$N(A^T)$ left null space, null space of A^T

$C(A^T)$ row space of A

$N(A)$ null space of A

$$y = Ax \qquad A \in \mathbb{R}^{m \times n} \qquad x \in \mathbb{R}^n \quad y \in \mathbb{R}^m$$

$$C(A) \subseteq \mathbb{R}^m \qquad N(A^T) \subseteq \mathbb{R}^m \qquad \mathbb{R}^m = C(A) + N(A^T)$$

$$C(A^T) \subseteq \mathbb{R}^n \qquad N(A) \subseteq \mathbb{R}^n \qquad \mathbb{R}^n = C(A^T) + N(A)$$