FTLA: concepts and statement

Monday, May 22, 2023 11:41 AM

- $A \in \mathbb{R}^{m \times n}$ (matrix) $\quad A = [a_1 \ a_2 \ \dots \ a_n]$
- $f: \mathbb{R}^n \to \mathbb{R}^m$ linear mapping

  $\forall u, v \in \mathbb{R}^n \quad \forall x, y \in \mathbb{R}$

  $$f(xu + yv) = x f(u) + y f(v)$$

- Assertion: $\quad f(u) = Au \quad f: \mathbb{R}^n \to \mathbb{R}^m$

  $u \in \mathbb{R}^n \quad f(u) \in \mathbb{R}^m$

  f is a linear mapping

$$f(xu + yv) = A(xu + yv)$$

matrix vector scalar

$$f(xu + yv) = A(xu + yv)$$

$$x f(u) = xAu \qquad y f(v) = yAv$$

$$A(xu + yv) = xAu + yAv$$

Matrix-vector multiplication $\quad A(s+t) = As + At$

$$A(xu+yv) = (Axu + Ayv) =$$
$$= xAu + yAv = xf(u) + yf(v).$$

How do we get matrices for specific linear mappings?

$f : \mathbb{R}^n \to \mathbb{R}^m$; f linear mapping

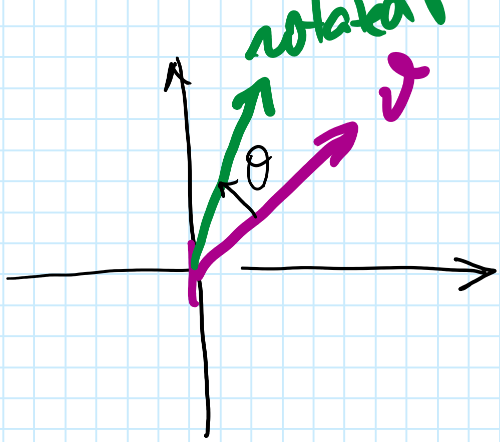
$I_n \in \mathbb{R}^{n\times n}$ $\quad I_n = [e_1\ e_2 \dots e_n]$

$$e_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\ e_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \dots,\ e_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$

The matrix representing f is

$$A = [f(e_1)\ \ f(e_2)\ \dots\ f(e_n)]$$

Ex: Rotation in $\mathbb{R}^2$



Q: Is rotation a linear mapping

Question: Is rotation a linear mapping

$f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad m=2\ ,\ n=2$

$f(xu + yv) = x f(u) + y f(v)$

$\forall\, x, y \in \mathbb{R} \quad \forall\, u, v \in \mathbb{R}^2$

$f(xu) = x f(u)$

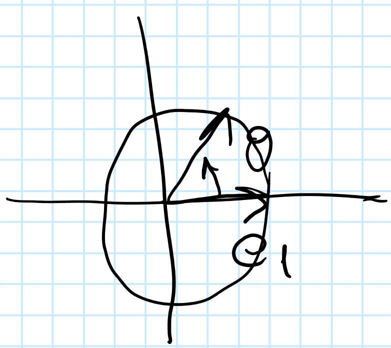
$f(xu)$ = rotation of u scaled by x ✓

$x f(u)$ = rotated u then scaled by

$f(u+v) = f(u) + f(v)$ ✓

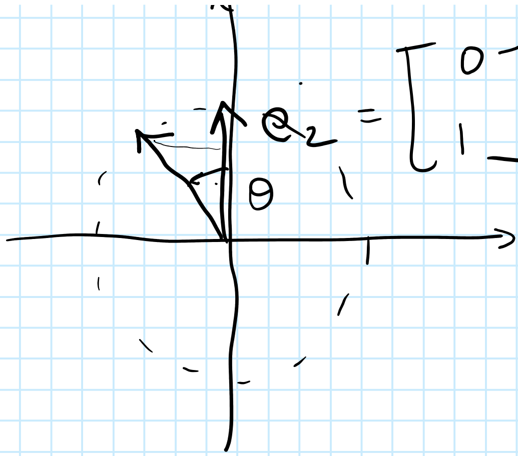
$\Rightarrow$ Indeed rotation is a linear mapping

$A = [\, f(e_1) \ \ f(e_2) \,] \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$f(e_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

$f(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$



$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$; $f(e_2) = \begin{bmatrix} \ \\ \cos\theta \end{bmatrix}$

$$A = [f(e_1) \ \ f(e_2)] = \begin{array}{cc} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ \begin{array}{cc} f(e_1) & f(e_2) \end{array} \end{array}$$

Apply to rotation of v

"Rotated v" $= f(v) = Av$

"rotated v" $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$= \begin{bmatrix} v_1 \cos\theta - v_2 \sin\theta \\ v_1 \sin\theta + v_2 \cos\theta \end{bmatrix}$$

<u>Ex 2</u>: Stretching

- stretch by factor α along the first direction "the e_1 direction" first direction "the e_1 direction"

• — " — β — " —

second — " — "the e_2 direction"

$f: \mathbb{R}^2 \to \mathbb{R}^2$

Question is f a linear mapping

$$f(\alpha u) = \alpha f(u)$$

$$f(u+v) = f(u) + f(v)$$

What is the matrix representing stretching?

$$A = [f(e_1) \; f(e_2)]$$

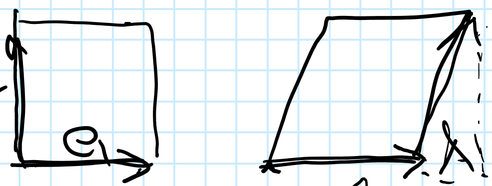
$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad f(e_1) = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad f(e_2) = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

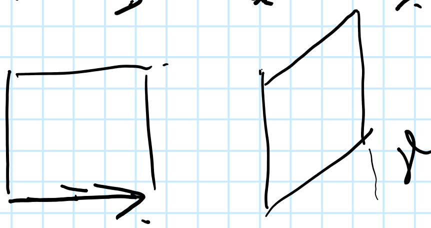
$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

Ex 3: Shearing

a) f



b) g



$\alpha \beta \gamma \delta \varepsilon \ldots \lambda$

$f: \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad$ $g: \mathbb{R}^2 \to \mathbb{R}^2$

$f(u) = Au$ $\qquad$ $g(u) = Bu$

$$A = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ \gamma & 1 \end{bmatrix}$$

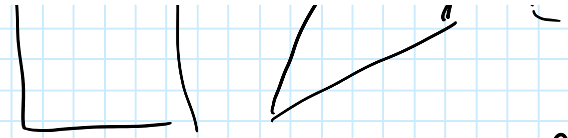
A is shearing along the "second dir" by λ

B — " — first by γ

Question: What happens if shearing occurs in both directions?

$h: \mathbb{R}^2 \to \mathbb{R}^2$

=



"first shear along y" + "then shear along x"

$$h(u) = \underline{g}(\underline{f}(u)) = (g \circ f)(u)$$

$$h(u) = g(Au) = BAu = Cu$$

$$C = BA$$

Ex:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

f rotation by α : $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

g rotation by β $\quad B = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

h rotation by $\alpha+\beta$

$$C = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \end{bmatrix}$$

$$C = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

$$C = BA$$

$$f : \mathbb{R}^2 \to \mathbb{R}^2 \quad f(u) = Au$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2 \quad g(u) = Bu$$

$$h(u) = (g \circ f)(u) = g(f(u)) =$$

$$h(u) = g(Au) = BAu$$

$$\begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} =$$

$$\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & \\ & \end{bmatrix}$$

$$\begin{bmatrix} \\ \cos\alpha\sin\beta + \cos\beta\sin\alpha \end{bmatrix}$$

Ex: Rotations in 3D

What are the rotation matrices in 3D?

$f: \mathbb{R}^3 \to \mathbb{R}^3$ rotation along x_1-axis by α

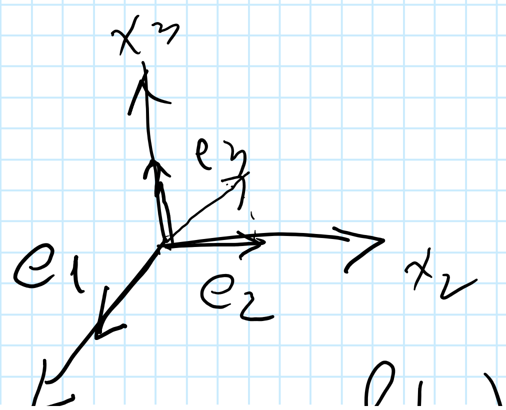
g — " — " — x_2 — " — β

h — " — " — x_3 — by γ

f, g, h are linear mappings $\Rightarrow$

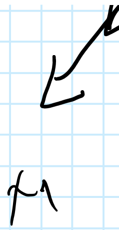
$$f(u) = Au \quad g(v) = Bv \quad h(w) = Cw$$

$$A = [\,f(e_1) \;\; f(e_2) \;\; f(e_3)\,]$$



$$f(e_1) = f\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$f(e_2) = f\left(\begin{bmatrix}0 \\ \end{bmatrix}\right) = \begin{bmatrix}0 \\ \end{bmatrix}$$



$$f(e_2) = f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$f(e_3) = f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{bmatrix}$$

$$A = [f(e_1)\ f(e_2)\ f(e_3)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$B = [g(e_1)\ g(e_2)\ g(e_3)]$$

$$= \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix}$$

(Insightful answer Santoshni +3 credit pts)

$$C = [h(e_1)\ h(e_2)\ h(e_3)] = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Combined rotation: first by α along x_1
then " β " x_2
" — γ along x_3

$$h(g(f(u))) = (h \circ g \circ f)(u) = \ell(u)$$

$$\ell(u) = Du$$

$$D = C \cdot B \cdot A$$

$f(u) = Au$
$g(v) = Bv$
$h(w) = Cw$

$$\ell(u) = h(g(Au)) = h(BAu) = CBAu$$

$$\overset{\shortparallel}{Du} = CBAu \Rightarrow D = CBA$$

Discussion of significance of column space, null space

Ex: $C(A) = \mathbb{R}^2 \quad N(A^T) = \{0\}$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$f : \mathbb{R}^2 \to \mathbb{R}^2$ represented by A

$\ell(u) = Au$

$\sin\theta \quad \cos\theta \,]$ $\quad f(u) = Au$

codomain

$$\mathbb{R}^2 = \underset{\mathbb{R}^2}{\underset{\shortparallel}{C(A)}} \oplus \{0\}$$

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad g(u) = Au$

$$C(A) = \{ y \mid \exists x \quad y = Ax \}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad y = Ax = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$C(A) = C\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$N(A^T) = \{ x \mid A^T x = 0 \}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \quad A^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \cdots \\ 0 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} \cdots \\ 0 & 0 \end{bmatrix}$$

$$A^T x = A^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

$$A^T x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x_1 = 0$$

$$N(A^T) = C\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad C(A) = C(e_1)$$

$$N(A^T) = C(e_2)$$

$$C(A) \oplus N(A^T) = \mathbb{R}^2$$

(Caitlyn +3)