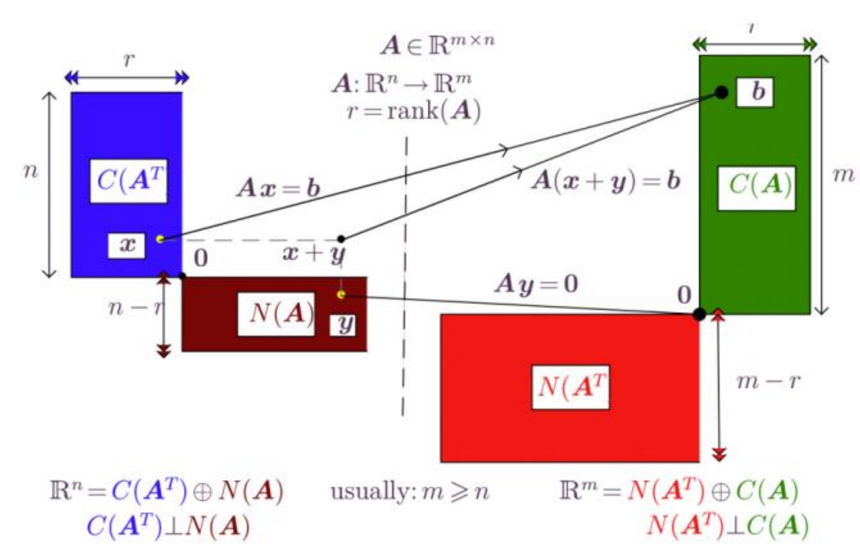
FTLA - associated definitions

Tuesday, May 23, 2023 11:34 AM

$r = \text{rank}(A) =$

$\dim C(A) = \dim C(A^T)$



Linear combination $\quad b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n \quad b, a_1, \dots, a_n \in \mathbb{R}^m$

$x_1, \dots, x_n \in \mathbb{R}$

$$b = Ax$$

$$A = [a_1 \ a_2 \ \dots \ a_n] \in \mathbb{R}^{m\times n}$$

$a_1, \dots, a_n$ column vectors; $a_1^T, \dots, a_n^T$ row vectors

$$a_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad a_1^T = [a_{11} \ a_{21} \ \dots \ a_{m1}]$$

Transpose

$$b = x_1 a_1 + \dots + x_n a_n \qquad b = Ax$$

$$b^T = x_1 a_1^T + \dots + x_n a_n^T \qquad ? \ \cancel{b^T = A^T x}$$

$$A = [a_1 \ a_2 \ \dots a_n] \qquad A^T = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$A \in \mathbb{R}^{m\times n}$

$A^T \in \mathbb{R}^{n\times m}$

$x \in \mathbb{R}^n$

$\cancel{A^T x}$

$$A^T x$$

$$\begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$b^T = x_1 a_1^T + \dots + x_n a_n^T \qquad A^T = \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix}$$

$$b^T = \underline{[x_1 \ \dots \ x_n]} \begin{bmatrix} a_1^T \\ \vdots \\ a_n^T \end{bmatrix} = x_1 a_1^T + \dots + x_n a_n^T$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{b^T = x^T A^T}$$

Recap: $\underline{b} = \underline{A}\underline{x} \Rightarrow b^T = x^T A^T$

Suppose we want to do ~~a lot~~ p of linear combinations

$$\boxed{A \in \mathbb{R}^{m \times n}} \qquad A = [a_1 \ \dots \ a_n]$$

$$u_1 \in \mathbb{R}^m \qquad \underline{v_1 \in \mathbb{R}^n}$$

$$\begin{cases} u_1 = A v_1 \\ u_2 = A v_2 \\ \dots \\ u_p = A v_p \end{cases}$$

$U = [u_1 \ u_2 \ \dots \ u_p]$ juxtapose all the linear combinations

$$U \in \mathbb{R}^{m \times p} \qquad A \in \mathbb{R}^{m \times n}$$

$$V = [v_1 \ v_2 \ \dots \ v_p] \qquad V \in \mathbb{R}^{n \times p}$$

$$V = [v_1 \;\; v_2 \;\dots\; v_p] \qquad V \in \mathbb{R}^{n \times p}$$

$$U = AV$$

Matrix-matrix multiplication = doing many linear combinations at once

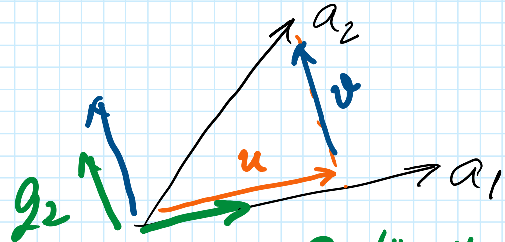
Given matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$ the matrix product is

$$C = AB, \quad C \in \mathbb{R}^{m \times p}$$

— $b = Ax \Rightarrow \quad b^T = x^T A^T$

$$C^T = (AB)^T = B^T A^T$$

— Gram-Schmidt



$$u = \underbrace{(q_1^T a_2)}_{\text{scalar}} q_1$$

$$q_1 = a_1 / \|a_1\|$$

$$a_2 = u + v$$

$$v = a_2 - u$$

$$v = a_2 - (q_1^T a_2) q_1$$

$$q_2 = v / \|v\|$$

Ex1: $A = [a_1 \;\, a_2] \in \mathbb{R}^2 \qquad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad a_1^T a_2 = 1$

$$q_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad v = a_2 - (q_1^T a_2) q_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - (1) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Ex2: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$v = \begin{bmatrix} 1 & \dots \end{bmatrix} \dots \begin{bmatrix} 1 & \dots \end{bmatrix} \begin{bmatrix} 1 & \dots \end{bmatrix} \begin{bmatrix} 1/2 & \dots \end{bmatrix} = \begin{bmatrix} 1/2 & \dots \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$\|v\| = \left(\frac{1}{4} + \frac{1}{4} \right)^{\frac{1}{2}} = \left(\frac{1}{2} \right)^{1/2} = \frac{1}{\sqrt{2}}$$

$$q_2 = \frac{1}{\|v\|} v = \sqrt{2} \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Ex 3

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \quad q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}; \quad q_2 = \begin{bmatrix} \frac{\sqrt{2}}{2\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{2}}{2\sqrt{3}} \end{bmatrix}$$

$$\|q_2\| = \frac{2}{4 \cdot 3} + \frac{2}{3} + \frac{2}{4 \cdot 3} = \frac{2}{3} \left(\frac{1}{4} + 1 + \frac{1}{4} \right) =$$

$$= \frac{2}{3} \frac{6}{4} = \frac{2}{3} \cdot \frac{3}{2} = 1 \quad \checkmark \text{ (E.C)}$$

$$q_1^T q_2 = 0$$

Ex.4

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad q_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} a_1$$

$$q_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} a_2 \quad q_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} a_3$$

$$A = [a_1 \ a_2 \ a_3]$$