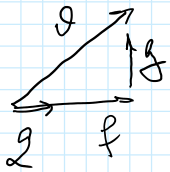
Projection, linear system solutions LSQ

Tuesday, May 23, 2023 11:34 AM

$Q = [q]$



$\|q\| = 1$ $\quad v = f + g$

$$f = (q^T v) q = q (q^T v) = (q q^T) v = P_q v$$

$$g = v - f = v - q q^T v = I \cdot v - q q^T v = (I - q q^T) v = P_{\perp q} v$$

Projector onto $C(Q)$ $\quad Q = [q_1\ q_2 \cdots q_n]$

$$\boxed{P_{C(Q)} = Q Q^T} = [q_1\ q_2 \cdots q_n] \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} = q_1 q_1^T + q_2 q_2^T + \cdots + q_n q_n^T$$

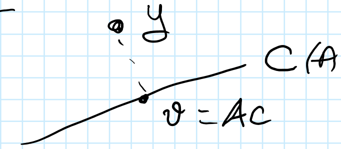
Least squares $\quad \mathcal{D} = \{(x_i, y_i), i = 1, \dots, m\}$ $\quad x, y \in \mathbb{R}^m$

"Proposed model" 1) Linear $y(x) = c_0 + c_1 x = c_0 \cdot 1 + c_1 x = c_0 \cdot x^0 + c_1 x^1$

Recognize a linear combination of $1, x$ with scaling coefficients c_0, c_1

Write linear combination as $\underbrace{[1\ \ x]}_{A} \underbrace{\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}}_{c} = Ac$

$$\min_{c \in \mathbb{R}^2} \| \overbrace{y - Ac}^{\text{error}} \|$$



v "best approximant" = orthogonal projection onto $C(A)$

$A = QR;$ $\quad v = P_Q y = Q Q^T y = Q R c \Rightarrow Q^T y = Rc$

$$\Rightarrow c = R \backslash Q^T y$$

2) $y(x) = c_0 \cdot 1 + c_1 x + c_2 x^2 = c_0 \cdot x^0 + c_1 x^1 + c_2 x^2$

$A = [1\ \ x\ \ x^2]$

$QR = A$ $\quad v = P_Q y = Q Q^T y = Q R c \Rightarrow$

$$c = R \backslash Q^T y$$

3) $y(x) = c_0 \cdot 1 + c_1 x + c_2 x^2 + c_3 x^3$

– the same –

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\- the same -

4) $y(x) = c_0 \cdot 1 + c_1 \cdot x + c_2 x^2 + c_4 \sin x$

$$A = [1 \quad x \quad x^2 \quad \sin x]$$

Gaussian elimination

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -l_{21} & 1 & & \\ -l_{31} & 0 & & \\ \vdots & \vdots & & \\ -l_{m1} & 0 & & \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ a_{31} & a_{32} & \\ \vdots & & \\ a_{m1} & a_{m2} & \end{pmatrix} = \begin{pmatrix} \boxed{a_{11}} & a_{12} & \cdots \\ \square & & \\ \square & & \end{pmatrix}$$

$$-l_{21} \cdot a_{11} + a_{21} = 0 \Rightarrow$$

$$l_{21} = \frac{a_{21}}{a_{11}}$$

$$-l_{31} \cdot a_{11} + a_{31} = 0 \Rightarrow$$

$$l_{31} = \frac{a_{31}}{a_{11}}$$

Do not work with individual components

"Don't see the forest for the trees"

$$L_{m-1} \cdots L_2 L_1 A = U$$

$2x = 4 \qquad x = 2 \qquad 2^{-1} = \frac{1}{\;}$

$$2x = 4 \qquad x = 2 \qquad 2^{-1} = \frac{1}{2}$$

$$2^{-1}(2x) = 2^{-1}(4)$$

$$(2^{-1}\cdot 2)x = 2^{-1}4 \Rightarrow \quad x = 2$$

$$ax = b \qquad x = a^{-1}b$$

$$Ax = b \qquad x = A^{-1}b$$

$$A^{-1}(Ax) = A^{-1}b$$

$$\left.\begin{array}{l}(A^{-1}A)x = A^{-1}b \\ A^{-1}A = I\end{array}\right| \Rightarrow \quad x = A^{-1}b$$

$$L_k = \begin{array}{c} \quad\quad k \\ \begin{pmatrix} 1 & & & & \\ 0 & \ddots & & & 0 \\ \vdots & & 1 & & \\ 0 & & -l_{k+1,k} & 1 & \\ & & \vdots & & \ddots \\ & & -l_{m,k} & & & 1 \end{pmatrix} k \end{array}$$

$$L_k^{-1} = \begin{pmatrix} 1 & & & & \\ 0 & \ddots & 1 & & \\ \vdots & & l_{k+1,k} & \ddots & \\ 0 & & \vdots & & \\ & & l_{m,k} & & 1 \end{pmatrix}$$

$$L_k^{-1} L_k = I$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & \vdots & & \\ 0 & 0 & & 1 & & \\ \vdots & \vdots & & l_{k+1,k} & & \\ 0 & 0 & & \vdots & & \\ & & & l_{m,k} & & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & 1 & & & & \\ \vdots & & \ddots & 1 & & \\ \vdots & & & -l_{k+1,k} & & \\ 0 & & & \vdots & & \\ & & & -l_{m,k} & & \end{pmatrix} =$$

$$= I$$

$= I$

Gaussian Elimination

$$L_{m-1} \cdots L_1 A = U$$

$$L_{m-1}^{-1} (L_{m-1} \cdots L_1) A = L_{m-1}^{-1} U$$

$$(L_{m-2} \cdots L_1) A = L_{m-1}^{-1} U$$

$$L_{m-2}^{-1} (L_{m-2} \cdots L_1) A = L_{m-2}^{-1} L_{m-1}^{-1} U$$

$$A = L_1^{-1} \cdot L_2^{-1} \cdots L_{m-1}^{-1} U$$

$$= L U$$

Gaussian elimination is another type of factorization

$A = QR$

Q orthogonal

R upper triangular

$A = LU$

L lower triangular

U upper triangular

Algorithm $\quad Ax = b \quad A \in \mathbb{R}^{m \times m}$

Given A, b

$L = I \quad U = I$

for $s = 1$ to $m-1$

  for $i = s+1$ to m

    $l_{is} = -a_{is} / a_{ss}$

$l_{is} = -a_{is}/a_{ss}$

for $j = s+1$ to m

$a_{ij} = a_{ij} + l_{is}\, a_{sj}$

end

end

end

Octave / Matlab

[L U] = lu(A)

[Q R] = qr(A,0)

$\Rightarrow$ LU costs $O\left(\frac{m^3}{3}\right)$ operations

	LSQ (QR factorization)	Linear syst. solved with LU
Problem	$\min_{x \in \mathbb{R}^n} \lVert y - Ax \rVert,\ A \in \mathbb{R}^{m\times n}$	$Ax = b \qquad A \in \mathbb{R}^{m\times m}$
	$QR = A$ $\quad m^3$	$LU = A \quad \frac{m^3}{3}$
	$P_Q y = QQ^T y = QRx$	$(LU)x = b$
	$\Rightarrow Rx = Q^T y$	$\frac{m^2}{2}\ Ly = b$ Forward substitution
	$x = R \setminus Q^T y \quad \frac{m^2}{2}$	$\frac{m^2}{2}\ Ux = y$ Backward substitution
	$(m=n)$	
	$m^3 + \frac{m^2}{2}$	$\frac{m^3}{3} + m^2$