Echelon forms, determinants

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$A \in \mathbb{R}^{m \times n}$ $\quad A = [a_1 \ a_2 \ a_3 \ \dots \ a_n]$ $\quad a_1, \dots, a_n \in \mathbb{R}^m$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

label of the column vector

label of the row vector

Basic operation in Gauss elimination is row linear combination

scalar (row i) + (row j) → obtain a new matrix

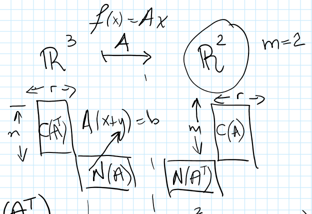
$\cancel{X}$

0

$$A \sim \begin{bmatrix} 1 & x & x & x & \dots & x \\ 0 & 1 & & & & \\ \vdots & & & & & \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & & & \end{bmatrix}$$

x = some non-zero value

$f(x) = Ax$



$\mathbb{R}^2 = C(A) \oplus N(A^T)$

$2 = \dim C(A) + \dim N(A^T)$

Ex: $\quad A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$

$\operatorname{rank} A = \dim C(A) = \dim C(A^T)$

$\operatorname{rank}(A) \leq \min(\dim C(A), \dim C(A^T))$

$\mathbb{R}^3 = C(A^T) \oplus N(A)$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

"echelon form" "reduced row echelon"

"similar" has same fundamental spaces

rank(A) = nr. of ones on diagonal of r.r.e.f.

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

Write the original system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = b_1 \\ -x_1 + x_3 = b_2 \end{cases}$$

The transformed system

$$\begin{cases} x_1 - x_3 = c_1 \\ x_2 + 2x_3 = c_2 \end{cases}$$

x_3 as a degree-of-freedom (it can take any value & the system will have a solution)

$$\begin{cases} x_1 = c_1 + x_3 \\ x_2 = c_2 - 2x_3 \end{cases}$$

$\underbrace{\dim N(A)}_{= 1} + \underbrace{\dim C(A^T)}_{= \text{nr. of ones in r.r.e.f.}} = n = 3$

$$\begin{cases} x_1 - x_3 = 3 \\ x_2 + 2x_3 = 4 \end{cases}$$

for $x_3 = 0$ $\quad \begin{cases} x_1 = 3 \\ x_2 = 4 \end{cases}$

for $x_3 = 1$ $\quad \begin{cases} x_1 = 4 \\ x_2 = 2 \end{cases}$

Ex:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$ $\quad \operatorname{rank}(A) = 1$

$\operatorname{nullity}(A) = 1$ = "nr. of zeros on diagonal"

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Write out the transformed

$x_1 + 2x_2 + 3x_3 = c_1 \qquad x_1 = c_1 - 2x_2 - 3x_3$

$0 = c_2$ for c_2

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \qquad f = Ax$

- $\mathbb{R}^n = C(A^T) \oplus N(A)$. $\qquad$ rank $A = 1$
- $\mathbb{R}^m = C(A) \oplus N(A^T)$

Nr. of columns $n = 3 = \dim C(A^T) + \dim N(A) = 1 + 2$

Nr. of rows $m = 2 = \dim C(A) + \dim N(A^T)$

Ex $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ -2 & 3 & 5 \\ 4 & 1 & -3 \end{bmatrix} \in \mathbb{R}^{4\times 3} \qquad f(x) = Ax \qquad m = 4, \; n = 3$

$\mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$\mathbb{R}^n = \mathbb{R}^3 = \text{domain} = C(A^T) \oplus N(A)$

$\mathbb{R}^m = \mathbb{R}^4 = \text{codomain} = C(A) \oplus N(A^T)$

$\dim C(A) = \dim C(A^T) = \text{rank}(A)$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ -2 & 3 & 5 \\ 4 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & 7 & 7 \\ 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2 = \dim C(A) = \dim C(A^T)$

$\text{nullity}(A) = 1$

$$\begin{aligned} n = 3 = \dim \mathbb{R}^3 &= \dim C(A^T) + \dim N(A) \\ &= \text{rank}(A) + \text{nullity}(A) \\ &= 2 + 1 \end{aligned}$$

$\dim N(A^T) = ?$

$$\begin{aligned} m = 4 = \dim \mathbb{R}^4 &= \dim C(A) + \dim N(A^T) \\ &= \text{rank}(A) + \dim N(A^T) \\ &= 2 + 2. \end{aligned}$$

What is this good for?

What is this good for?

Ex: $Ax=b$ $\quad b \in C(A) \Rightarrow Ax=b$ has a solution

nullity(A) $= \dim N(A) = 0 \Rightarrow$ solution is unique

nullity$(A) = k \Rightarrow$ solution is not unique, there are infinitely many solutions with k-parameters

$b \notin C(A) \Rightarrow Ax=b$ does not have a solution

Ex: $Ax = 0$ $\quad$ clearly $x=0$ is a solution

In $\mathbb{R}$-arithmetic $u \cdot v = 0 \Rightarrow$ at least one factor is zero

In lin. alg. $Ax=0 \not\Rightarrow$ that either $x=0$ or $A=0$

Def Square matrices $A \in \mathbb{R}^{m\times m}$ that have nullity$(A)>0$ are said to be singular.

Determinants

$A \in \mathbb{R}^{m\times m}$

- algebraic definition
- geometric definition

Algebraic: $\sigma : \begin{pmatrix} 1 & 2 & 3 & \dots & m \\ i_1 & i_2 & i_3 & \dots & i_m \end{pmatrix}$ $\quad i_1, \dots, i_m \in \{1,2,3,\dots,m\}$

$i_j \neq i_k$ for $j \neq k$

How many permutations of m objects

$\begin{pmatrix} 1 & 2 & 3 & \dots & m \\ \square & \square & & & \end{pmatrix}$ $\quad m\cdot(m-1)\cdot\ldots\cdot 1 = m!$

m choices

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ 1 "swap" of a pair $\rightarrow$ odd perm (-1)

$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ 2 "swaps" of pairs $\rightarrow$ even perm $(+1)$

$\nu(\sigma) =$ sign of permutation $= \begin{cases} +1 & \text{even perm} \\ -1 & \text{odd perm.} \end{cases}$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{vmatrix} = \sum_{\sigma} \underline{\nu(\sigma)}\ \underline{a_{1i_1} \cdots a_{mi_m}}$$

$$\sigma : \begin{pmatrix} 1 & 2 & & m \\ i_1 & i_2 & \cdots & i_m \end{pmatrix}$$

That's the rigorous definition

By definition $m \cdot m!$ FLOPS are required to compute a determinant.

$\underline{m=2}$ $\quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\qquad \underset{\underline{\sigma_1}}{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}} \quad \underset{\underline{\sigma_2}}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{12} a_{21}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

$m=3$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

sign of perm

$=$

$1\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad -1\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \quad -1\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$

$-1\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad 1\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad 1\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$$\det A = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{12} a_{21} a_{33} - a_{11} a_{23} a_{32} - a_{13} a_{22} a_{31}$$

Sarrus rule

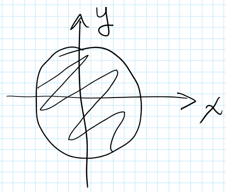
$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$\begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{matrix}$$

$a_{11} a_{22} a_{33} +$
$a_{21} a_{32} a_{13} +$
$a_{31} a_{12} a_{23}$
$- a_{13} a_{22} a_{31}$
$- a_{23} a_{32} a_{11}$
$-$ [illegible]

[illegible]

$$- a_{23}\,a_{32}\,a_{11}$$
$$- a_{33}\,a_{12}\,a_{21}$$

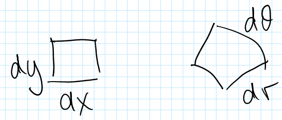
Ex.: though not widely applicable, the above are sometimes useful



$$A = \iint_{\text{circle}} dx\,dy$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dx\,dy = \frac{D(x,y)}{D(r,\theta)}\,dr\,d\theta = J\,dr\,d\theta$$



J = Jacobian determinant

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$A = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = \frac{1}{2}R^2\cdot 2\pi = \pi R^2.$$

Ex.: ODEs linear system of ODEs $y_1(t), \dots, y_m(t)$

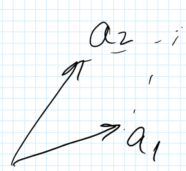
$$\begin{cases} y_1' = a_{11}y_1 + \dots + a_{1m}y_m \\ \vdots \\ y_m' = a_{m1}y_1 + \dots + a_{mm}y_m \end{cases} \qquad y = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

$y' = Ay \quad \Rightarrow \det A \Rightarrow$ information on solutions

$\rightarrow$ MATH 383

Geometric definition $A \in \mathbb{R}^{m\times m}$

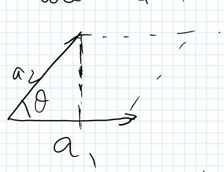
$A = [a_1 \ a_2 \ \dots \ a_m]$



$\det A = |A| = $ (signed) volume

"hyperparallelipiped"

$m = 2 \Rightarrow$ area of the parallelogram



$$\text{Area} = \|a_1\|\,\|a_2\|\sin\theta$$

$$\text{Projector onto } a_1 = P_1 = \frac{a_1 a_1^T}{\|a_1\|\,\|a_1\|}$$

$$P_1 \overset{?}{=} a_1^T \cancel{a_2}\, a_1$$

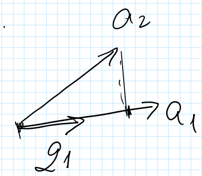
Complementary projector =

$$P_{1\perp} = I - P_1 = I - \frac{a_1 a_1^T}{\|a_1\|\,\|a_1\|}$$

$P_1 \overset{?}{=} a_1^T \cancel{a_2}\, a_1$

Complementary projector = $P_\perp = I - P_1 = I - \frac{a_1 a_1^T}{\|a_1\| \|a_1\|}$

Recall



$q_1 = \frac{a_1}{\|a_1\|}$

Area $= \|a_1\| \, \|P_\perp a_2\|$

$= \|a_1\| \, \| a_2 - \frac{a_1 a_1^T a_2}{\|a_1\|^2} \| = a_{11}a_{22} - a_{12}a_{21}$

Part of a_2 in the direction of a_1

= projection of a_2 in the direction of a_1

$= (q_1^T a_2) q_1 = q_1 (q_1^T a_2) =$

$= (q_1 q_1^T) a_2 = P_1 a_2 = v$

projector

projection = result of applying the projector

$P_1 = q_1 q_1^T = \frac{a_1 a_1^T}{\|a_1\|^2}$

Determinant of A = volume of body bounded by the columns of A.

Proposition: Gaussian elimination "(scalar) row added to a row" do not change the value of a determinant

Ex:
$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 1 & 0 & -1 & 1 \\ -2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

$$\det A = \begin{vmatrix} 4 & 2 & 1 & 3 \\ 1 & 0 & -1 & 1 \\ -2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & -1 & 1 \\ 4 & 2 & 1 & 3 \\ -2 & 3 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & 5 & -1 \\ 0 & 3 & -1 & 4 \\ 0 & 0 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -1 \\ 3 & -1 & 4 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 5 & -1 \\ 0 & -\frac{17}{2} & \end{vmatrix} = \ldots = 22$$

$$= \begin{vmatrix} 2 & 3 & 1 \\ 0 & -\frac{17}{2} & \\ 0 & 2 & 0 \end{vmatrix} = \dots = 22$$

$-1 - 5\frac{3}{2} =$

$$\boxed{\begin{array}{c} A \text{ singular iff } \det(A) = 0 \Rightarrow \\ A \text{ has linearly dependent columns} \Rightarrow \\ \dim N(A) > 0 \end{array}} \rightarrow$$

Eigenproblems

<u>Statement</u> For square matrix $A \in \mathbb{R}^{m \times m}$ find $x \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}$ that satisfy

$$Ax = \lambda x \qquad \lambda = \text{eigenvalues}$$
$$x \neq \vec{0} \qquad x = \text{eigenvectors}$$

Rewrite $Ax = \lambda x$ as $Ax = \lambda I \cdot x \Rightarrow$

$$Ax - \lambda I x = 0 \Rightarrow$$
$$\underbrace{(A - \lambda I)}_{\text{matrix}} x = 0$$

$\Rightarrow$ $A - \lambda I$ is singular $\dim N(A - \lambda I) > 0$

$$\det(A - \lambda I) = 0 \Rightarrow$$
$$\det(\lambda I - A) = 0$$

$$p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \dots & -a_{1m} \\ -a_{21} & \lambda - a_{22} & \dots & -a_{2m} \\ & & & \\ \vdots & & & \end{vmatrix}$$

$P_A(\lambda)$ characteristic polynomial of A of degree n

Ex: $A \in \mathbb{R}^{2\times 2}$ $\quad P_A(\lambda) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} =$

$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21}$

Does A have eigenvalues / eigenvectors? Yes, n eigenvalues

$$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0, \quad x \neq 0 \quad \Rightarrow$$

$$A - \lambda I \text{ singular} \Rightarrow \det(A - \lambda I) = 0 \Rightarrow$$

$$\underline{P_A(\lambda) = 0} \rightarrow \text{roots of characteristic polynomial}$$

$\Rightarrow$ there are n such roots

Ex: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ What are the eigenvalues & eigenvectors of I

$$P_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 \Rightarrow$$

roots are $\lambda_1 = \lambda_2 = 1$ (a double root)

Eigenvectors satisfy $(A - \lambda I)x = 0$

$$\Rightarrow x \in N(A - \lambda I)$$

r.r.e.f $\quad A - \lambda I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has rank $= 0$, has nullity $= 2 \Rightarrow$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ has rank $= 0$, has nullity $= 2 \Rightarrow$

$$N(A - \lambda I) = \mathbb{R}^2$$

a basis for $\mathbb{R}^2$ is $\{e_1, e_2\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$I \cdot e_1 = 1 \cdot e_1 \qquad I \cdot e_2 = 1 \cdot e_2$

Ex: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$P_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{vmatrix} =$$

$$= (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda + 1 - 1 =$$

$$= \lambda(\lambda - 2) < \begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 2 \end{matrix}$$

$\lambda_1 = 0 \qquad A - \lambda I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{cases} y_1 + y_2 = 0 \\ 0 = 0 \end{cases} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_1 = 0 \quad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 2 \qquad A - \lambda_2 I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$

$\lambda_2 = 2 \qquad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0 \Rightarrow A - \lambda I$ singular

$x \neq 0$

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0 \Rightarrow A - \lambda I$ singular

$x \neq 0$

$$\det(A - \lambda I) = 0 \qquad (\dim N(A - \lambda I) > 0)$$

$$p_A(\lambda) = \det(\lambda I - A)$$ characteristic polynomial of A

Eigenvalues are roots of the characteristic polynomial

There are always n such roots.

For each root (eigenvalue) the associated eigenvector is a solution of the system $(A - \lambda I)x = 0$

The eigenvector is in the $N(A - \lambda I)$

Ex:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$p_A(\lambda) = \det \begin{vmatrix} \lambda - 1 & -1 \\ -1 & \lambda + 1 \end{vmatrix} = \lambda^2 - 1 - 1 = \lambda^2 - 2$$

$$\lambda_1 = \sqrt{2} \qquad \lambda_2 = -\sqrt{2}$$

$\lambda_1 = \sqrt{2}$; $(A - \lambda I)y = 0$

$$\begin{pmatrix} 1 - \sqrt{2} & 1 \\ 1 & -1 + \sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1 - \sqrt{2} & 1 \\ 0 & 0 \end{pmatrix}$$

$$(1 - \sqrt{2})y_1 + y_2 = 0 \qquad \begin{pmatrix} \frac{1}{\sqrt{2} - 1} \\ 1 \end{pmatrix} = x_1$$

$$y_1 = -\frac{y_2}{1 - \sqrt{2}}$$

$$\begin{pmatrix} 1 + \sqrt{2} & 1 \end{pmatrix}$$

$1-\sqrt{2}$

$$\lambda_2 = -\sqrt{2} \qquad \begin{pmatrix} 1+\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{pmatrix} \sim \begin{pmatrix} 1+\sqrt{2} & 1 \\ 0 & 0 \end{pmatrix}$$

$$x = \begin{bmatrix} -\frac{1}{1+\sqrt{2}} \\ 1 \end{bmatrix}.$$