Eigendecomposition

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Brief recap $b \in \mathbb{R}^m$ (m large) $b = I \cdot b$ $I \in \mathbb{R}^{m \times m}$ m coordinates

$a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ ($n \ll m$) $b = Ax$ if $b \in C(A)$

find new coord. by solving linear system

find coordinates of the best approximation

by solving LSQ $\min_{x \in \mathbb{R}^n} \|b - Ax\|$ least squares

Note:

LSQ not written "left-to-right"

$\min_{x \in \mathbb{R}^n} \|b - Ax\|$

written "inside-out" $\min_{x} \|b - Ax\|$

Problems are solved by factorization

- $\underset{m \times n}{A} = \underset{m \times n}{Q}\,\underset{n \times n}{R}$ Q orthogonal $Q^T Q = I_n$

$P_{C(A)}$ = projector onto $C(A) = QQ^T$ $\left(v = P_{C(A)} b = QQ^T b \right.$

$v = Ax = QRx$

$QRx = QQ^T b \Rightarrow Rx = Q^T b$

x coordinates of best approx

- $A = LU$ (Gaussian elimination)

Eigenproblem: $A \in \mathbb{R}^{m \times m}$ find λ, x s.t. $x \neq 0$

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$

$P_A(\lambda) = |\lambda I - A|$ characteristic polynomial of degree m

λ_1: $Ax_1 = \lambda_1 x_1$ λ_2: $Ax_2 = \lambda_2 x_2$ ... λ_m $Ax_m = \lambda_m x_m$

X = eigenvector matrix $= [x_1 \ x_2 \ \ldots \ x_m] \in \mathbb{C}^{m \times m}$

$$AX = [Ax_1 \ Ax_2 \ \ldots \ Ax_m] = [\lambda_1 x_1 \ \lambda_2 x_2 \ \ldots \ \lambda_m x_m]$$

$$= [x_1 \ x_2 \ \ldots \ x_m] \begin{bmatrix} \lambda_1 & 0 & & \\ 0 & \lambda_2 & & \\ 0 & 0 & \ddots & \\ 0 & & & \end{bmatrix} = X\Lambda$$

Λ diagonal matrix

$$\begin{bmatrix} \vdots & 0 & & \\ \vdots & \vdots & 0 & \\ 0 & & \vdots & \end{bmatrix}$$

scalings of $x_1, \dots, x_m$ to obtain $\lambda_1 x_1$ by lin. comb.

Λ diagonal matrix

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{C}^{m\times m}$$

Matrix form of the eigenproblem $\quad AX = X\Lambda$

Question $\quad \cancel{AX = \Lambda X}$

Compare LU-factorization $\quad A = LU, \quad LU = A$

QR — " — $\quad A = QR$

Eigenproblem $\quad AX = X\Lambda \qquad A = X\Lambda X^{-1} \qquad X \neq 0$

(Scalar relation $\quad ax = x\lambda, \quad ax \cdot x^{-1} = x\lambda x^{-1} = \lambda$)
$x \neq 0$

X must be of full rank

columns of X, eigenvectors must be linearly independent

$\Rightarrow X$ is invertible & X^{-1} exists

Question: When is X invertible?

$A \in \mathbb{R}^{m\times m} \qquad p_A(\lambda) = \det(\lambda I - A) = |\lambda I - A|$ of degree m

& the coefficient of λ^m is 1 — standard

$$p_A(\lambda) = \lambda^m + c_{m-1}\lambda^{m-1} + \dots + c_1\lambda + c_0$$ ("canonical" form of a polynomial)

$$p_A(\lambda) = (\lambda - \lambda_1)(\lambda-\lambda_2)\cdot\dots\cdot(\lambda-\lambda_m)$$ factored form

$$x^2 - 1 = (x-1)(x+1)$$
$$(x-1)^2 = x^2 - 2x + 1$$

$\lambda_1, \lambda_2, \dots, \lambda_N$ are distinct eigenvalues

$n_1, n_2, \dots, n_N$ multiplicities of each root

"algebraic multiplicities"

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$$P_A(\lambda) = (\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_N)^{n_N}$$

$$n_1+n_2+\dots+n_N = m$$

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for some eigenvalue $\lambda_i \in \{\lambda_1,\dots,\lambda_N\}$

$\underline{E}_i = \underline{N}(A-\lambda_i I)$ eigenspace associated with eigenvalue λ_i

$$Ax = \lambda x \iff (A-\lambda I)x = 0 \qquad x \neq 0$$

$m_i = \dim E_i$ **geometric multiplicity**

if $\mathbb{C}^m = E_1 \oplus E_2 \oplus \dots \oplus E_N$ then X is invertible.

vector spaces

vector sums

i.e, $m = \sum_{i=1}^{N} m_i$ $\qquad 1 \le m_i \le m$

Conclusion: $AX = X\Lambda$ $\quad X$ is invertible if for all eigenvalues the algebraic multiplicity equals the geometric multiplicity.

The SVD

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ $\quad f$ linear mapping then $f(x) = Ax$

But the choice of basis made by specific choice of $A=[a_1 \dots a_n]$ might not be the most insightful.

The SVD $\quad \exists\, V \in \mathbb{R}^{n\times n}, V^TV = I_n, \quad \exists\, U \in \mathbb{R}^{m\times m}, U^TU = I_m$

$$\exists\, \Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \\ & & & & \ddots \end{bmatrix} \in \mathbb{R}_+^{m\times n} \qquad \sigma_1 \ge \sigma_2 \ge \dots \ge \sigma_r > 0$$

$r \le \min(m,n)$

$$\exists \ \Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_r & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} \in \mathbb{R}_{+}^{m \times n} \qquad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$$

$$r \leq \min(m, n)$$

s.t.

$$A = U \Sigma V^T$$