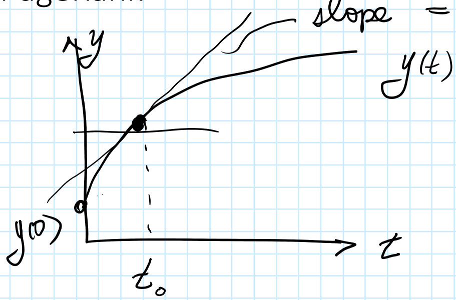
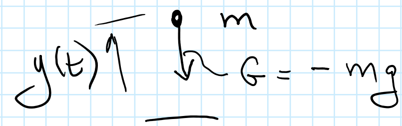
Model reduction, stochastic matrices, PageRank

Wednesday, May 31, 2023 11:37 AM

$y : \mathbb{R} \to \mathbb{R}$ $y(t)$

position along a trajectory





as $\Delta t \to 0$ slope becomes instantaneous velocity

$$\dot{y} = y' = \frac{dy}{dt} = v(t) \text{ velocity}$$

~~F = ma~~

Newton's 2nd law of dynamics

$$\frac{d}{dt}(m\dot{y}) = \underline{\sum \text{external forces}} = -mg$$

Does mass m vary in time? No

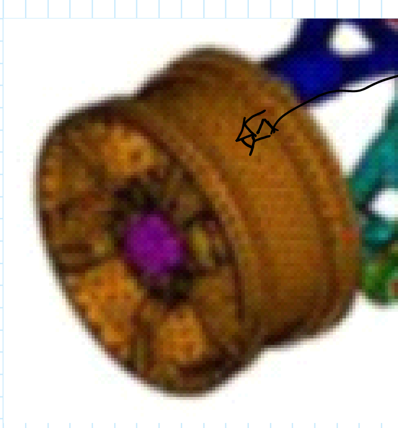
$$\frac{d}{dt}(m\dot{y}) = m\frac{d\dot{y}}{dt} = m\ddot{y} = -mg \Rightarrow$$

$$\ddot{y} = -g \quad \text{equation of motion}$$

Newton's 2nd law of dynamics

"Rate of change of momentum equals the sum of applied forces"

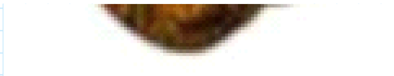
Momentum = (mass) × (velocity)



breaking down the object into small pieces

"finite element method"

Put together the positions of all finite elements into $y \in \mathbb{R}^m$



$y(t)$ finite element positions vary in time

$y: \mathbb{R} \to \mathbb{R}^m$

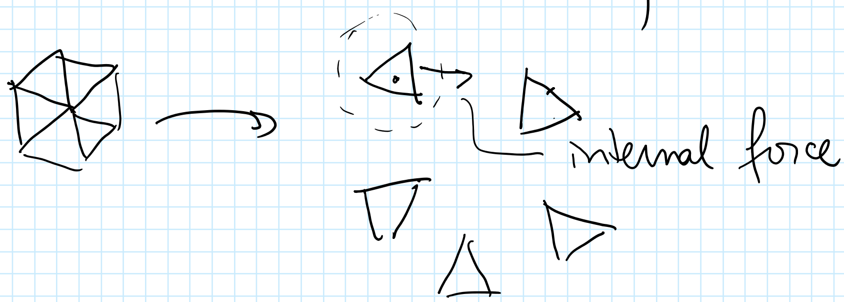
Single falling object	Suspension
$m\ddot{y} = -mg$	$M\ddot{y} = -Ky$

y → state

K → stiffness matrix

$\ddot{y}$ → acceleration

M → mass or inertia



Generic (it appears everywhere) problem

$$M\ddot{y} + Ky = 0$$

(linear mass-stiffness model)

m is large ; $m >> 1$ $\quad m \cong O(10^7 \text{ to } 10^9)$

$y \in \mathbb{R}^m$

Idea: Approximate y by a linear combination

$$y \cong Bx \qquad B \in \mathbb{R}^{m \times n}$$

$$n << m$$

→ $y(t)$ is approximated by a linear combination of basis functions $b_1(t), b_2(t), \ldots, b_n(t) \in \mathbb{R}^m$

$B(t) = [b_1(t) \; b_2(t) \; \ldots \; b_n(t)]$

$$B(t) = [b_1(t) \;\; b_2(t) \;\; \dots \;\; b_n(t)]$$

Equation of motion $\quad M\ddot{y} + Ky = 0.$ $\qquad \underline{y \cong Bx,}$

$$B^T B = I_n$$

$K \in \mathbb{R}^{m\times m}$ $\qquad y \in \mathbb{R}^m$

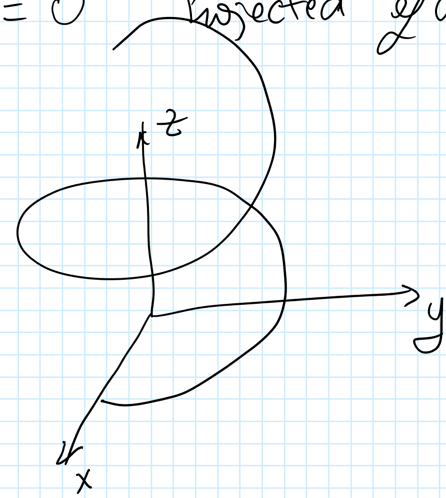
$$P_{C(B)} = BB^T$$

$$M\ddot{y} + Ky = 0 \qquad \text{Equation of motion}$$

$$BB^T(M\ddot{y} + Ky) = 0 \qquad \text{Projected equation of motion}$$

Ex:

$$\begin{cases} x(t) = r\cos t \\ y(t) = r\sin t \\ z(t) = t \end{cases}$$



$u = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\qquad$ projection onto x,y-plane $\rightarrow$ circle

Projection onto x-y plane

$$P = BB^T \qquad B = [e_1 \; e_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$P = BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ 0 \end{bmatrix}$$

Coming back to general theory

1) $y : \mathbb{R} \to \mathbb{R}^m \quad y(t) \quad m \gg 1$ — Model

$$M\ddot{y} + Ky = 0 \quad M, K \in \mathbb{R}^{m \times m}$$

2) $y \simeq B \cdot x \quad x : \mathbb{R} \to \mathbb{R}^n$ — Approximation

$$B \in \mathbb{R}^{m \times n} \quad \text{(orthogonal)}$$

3) $P_{C(B)} = BB^T$ projector — Project onto the approximation space

4) Projected equation of motion

$$BB^T (M\ddot{y} + Ky) = 0$$

$y = Bx$
$\dot{y} = B\dot{x}$
$\ddot{y} = B\ddot{x}$

$$BB^T M B\ddot{x} + BB^T K B x = 0$$

$$\underbrace{B}\,(\underline{B^T M B \ddot{x} + B^T K B x}) = 0$$

rank $B = n$

$$\Rightarrow \quad (\underbrace{B^T M B}_{\sim})\ddot{x} + (\underbrace{B^T K B}_{\sim})x = 0$$

"Reduced" equation of motion

$$\tilde{M}\ddot{x} + \tilde{K}x = 0$$

$\tilde{M}$ = reduced mass matrix $\in \mathbb{R}^{n \times n}$

$\tilde{M}$ = reduced mass matrix $\in \mathbb{R}$

$\tilde{K}$ = — " — stiffness matrix