## Overview

- Quantitites
- Vectors
- Vector operations
- Linear combinations
- Matrix vector multiplication
- Matrices
- Matrix-matrix multiplication

Numbers in mathematics

**$\mathbb{N}$.** The set of natural numbers, $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, infinite and countable, $\mathbb{N}_+ = \{1, 2, 3, \ldots\}$;

**$\mathbb{Z}$.** The set of integers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$, infinite and countable;

**$\mathbb{Q}$.** The set of rational numbers $\mathbb{Q} = \{p/q, p \in \mathbb{Z}, q \in \mathbb{N}_+\}$, infinite and countable;

**$\mathbb{R}$.** The set of real numbers, infinite, not countable, can be ordered;

**$\mathbb{C}$.** The set of complex numbers, $\mathbb{C} = \{x + iy, x, y \in \mathbb{R}\}$, infinite, not countable, cannot be ordered.

Numbers on a computer

**Subsets of $\mathbb{N}$.** The number types `uint8`, `uint16`, `uint32`, `uint64` represent subsets of the natural numbers (unsigned integers) using 8, 16, 32, 64 bits respectively.

**Subsets of $\mathbb{Z}$.** The number types `int8`, `int16`, `int32`, `int64` represent subsets of the integers. One bit is used to store the sign of the number.

**Subsets of $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.** Computers approximate the real numbers through the set $\mathbb{F}$ of *floating point numbers*. Floating point numbers that use $b = 32$ bits are known as *single precision*, while those that use $b = 64$ are *double precision*.

- Some quantities arising in applications can be expressed as single numbers, called "scalars"
  - Speed of a car on a highway $v = 35$ mph
  - A person's height $H = 183$ cm
- Many other quantitites require more than one number:
  - Position in a city: "Intersection of 86$^{th}$ St and 3$^{rd}$ Av"
  - Position in 3D space: $(x, y, z)$
  - Velocity in 3D space: $(u, v, w)$

**Definition.** *A* *vector* *is a grouping of* $m$ *scalars*

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} \in S^m, v_i \in S$$

- The scalars usually are naturals ($S = \mathbb{N}$), integers ($S = \mathbb{Z}$), rationals ($S = \mathbb{Q}$), reals ($S = \mathbb{R}$), or complex numbers ($S = \mathbb{C}$)
- We often denote the dimension and set of scalars as $\boldsymbol{v} \in S^m$, e.g. $\boldsymbol{v} \in \mathbb{R}^m$
- Sets of vectors are denoted as

$$\mathcal{V} = \{\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}, v_i \in S\} \tag{1}$$

- A vector can also be interpreted as a function from a subset of $\mathbb{N}$ to $S$

$$v\colon \{1, 2, .., m\} \to S$$

- **Vector addition**. Consider two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. We define the sum of the two vectors as the vector containing the sum of the components

$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

```
∴ u=[1 2 3]; v=[-2 1 2]; u+v
```

- **Scalar multiplication**. Consider $\alpha \in S$, $\boldsymbol{u} \in \mathcal{V}$. We define the multiplication of vector $\boldsymbol{u}$ by scalar $\alpha$ as the vector containing the product of each component of $\boldsymbol{u}$ with the scalar $\alpha$

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} = \begin{bmatrix} \alpha\ u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\, u_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

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$$\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_m + v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

```
∴ u=[1 2 3]; v=[-2 1 2]; u+v
```

$$\begin{bmatrix} -1 & 3 & 5 \end{bmatrix} \tag{2}$$

- **Scalar multiplication**. Consider $\alpha \in S$, $\boldsymbol{u} \in \mathcal{V}$. We define the multiplication of vector $\boldsymbol{u}$ by scalar $\alpha$ as the vector containing the product of each component of $\boldsymbol{u}$ with the scalar $\alpha$

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} = \begin{bmatrix} \alpha\ u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\, u_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

- **Linear combination**. Let $\alpha, \beta \in S$, $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$. Define a linear combination of two vectors by

$$\boldsymbol{w} = \alpha\ \boldsymbol{u} + \beta\ \boldsymbol{v} = \begin{bmatrix} \alpha\, u_1 \\ \alpha\ u_2 \\ \vdots \\ \alpha\ u_m \end{bmatrix} + \begin{bmatrix} \beta\ v_1 \\ \beta\ v_2 \\ \vdots \\ b\beta\, v_m \end{bmatrix} = \begin{bmatrix} \alpha\ u_1 + \beta\ v_1 \\ \alpha\ u_2 + \beta\ v_2 \\ \vdots \\ \alpha\, u_m + \beta\, v_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

- Linear combination of $n$ vectors

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \cdots + x_n \boldsymbol{a}_n = \begin{bmatrix} x_1 a_{11} + x_2 a_{12} + \cdots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \cdots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \cdots + x_n a_{mn} \end{bmatrix}$$

"Start at the center of town. Go east 3 blocks and north 2 blocks. What is your final position?"

$$\boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{e}_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\alpha_E = 3, \alpha_N = 2$$

$$\boldsymbol{f} = \boldsymbol{s} + \alpha_E \boldsymbol{e}_E + \alpha_N \boldsymbol{e}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Linear combinations allow us to express a position in space using a standard set of directions. Questions:

- How many standard directions are needed?
- Can any position be specified as a linear combination?
- How to find the scalars needed to express a position as a linear combination?

Seek a more compact notation for the linear combination

$$a\begin{bmatrix}1\\2\\3\end{bmatrix}+b\begin{bmatrix}-1\\0\\1\end{bmatrix}+c\begin{bmatrix}2\\0\\1\end{bmatrix}=\begin{bmatrix}a-b+2c\\2a\\3a+b+c\end{bmatrix}$$

- Group the vectors together to form a "matrix"

$$\boldsymbol{A}=\begin{bmatrix}1&-1&2\\2&0&0\\3&1&1\end{bmatrix}$$

- Group the scalars together to form a vector

$$\boldsymbol{u}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

- Define matrix-vector multiplication

$$\boldsymbol{A}\boldsymbol{u} = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a-b+2\,c \\ 2\,a \\ 3\,a+b+c \end{bmatrix}$$

- In general

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = [\ \boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \dots \quad \boldsymbol{a}_n\ ], \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \begin{bmatrix} x_1a_{11} + x_2a_{12} + \cdots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \cdots + x_na_{2n} \\ \vdots \\ x_1a_{m1} + x_2a_{m2} + \cdots + x_na_{mn} \end{bmatrix}$$

- Construct linear combination of vectors $\boldsymbol{u} = [\,1 \;\; -1 \;\; 2\,]$, $\boldsymbol{v} = [\,2 \;\; 1 \;\; -1\,]$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
```

```
∴ alpha*u+beta*v
```

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
```

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∴ u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
```

```
∴ alpha*u+beta*v
```

$$[\,8 \;\; 1 \;\; 1\,] \tag{3}$$

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
```

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1 -1 2]; v=[2 1 -1]; alpha=2; beta=3;
```

```
∴ alpha*u+beta*v
```

$$\begin{bmatrix} 8 & 1 & 1 \end{bmatrix} \tag{4}$$

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1; -1; 2]; v=[2; 1; -1]; alpha*u+beta*v
```

$$\begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} \tag{5}$$

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

```
∴ u=[1; -1; 2]; v=[2; 1; -1]; alpha=2; beta=3;
```

```
∴ A=[u v]; x=[alpha; beta]; A*x
```

- Construct linear combination of vectors $\boldsymbol{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ scaled by $\alpha = 2$ and $\beta = 3$, respectively

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```

```
∴ A=[u v]; x=[alpha; beta]; A*x
```

$$\begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix} \tag{6}$$

**Definition.** *An* $m$ *by* $n$ *matrix is a grouping of* $n$ *vectors,*

$$\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_n\ ] \in S^{m \times n},$$

*where each vector has* $m$ *scalar components* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in S^m$.

- Notation conventions:
  - scalars: normal face, Latin or Greek letters, $a, b, \alpha, \beta, u_1, a_{11}$, $A_{11}, I_{12}$
  - vectors: bold face, lower case Latin letters, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}_1$
  - matrices: bold face, upper case Latin letters, $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L}_1$

- Matrix components

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A real-valued matrix with $m$ lines and $n$: $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

- Matrix components

$$\boldsymbol{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \boldsymbol{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots, \boldsymbol{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \Rightarrow$$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A real-valued matrix with $m$ lines and $n$: $\boldsymbol{A} \in \mathbb{R}^{m \times n}$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{7}$$

- Instead of explicitly writing out components, it is often convenient to specify a matrix by a rule to construct each component

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}, \boldsymbol{A} = [a_{ij}]$$

with indices taking values $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.

Example: A *Hilbert matrix* $\boldsymbol{H}_m \in \mathbb{R}^{m \times m}$ is defined as $\boldsymbol{H}_m = \left[ \frac{1}{i+j-1} \right]$

```
∴ function hilb(m)
     H=ones(m,m)
     for i=1:m for j=1:m
                 H[i,j]=1.0/(i+j-1.0)
     end       end
     return H
  end;
```

```
∴ hilb(3)
```

- Note that a vector is a matrix with a single column. The notation $\boldsymbol{v} \in \mathbb{R}^m$, is a customary shorter form of $\boldsymbol{v} \in \mathbb{R}^{m \times 1}$.

- Instead of explicitly writing out components, it is often convenient to specify a matrix by a rule to construct each component

$$\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{A} = [a_{ij}]$$

with indices taking values $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}$.

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                 H[i,j]=1.0/(i+j-1.0)
     end       end
     return H
  end;
```

```
∴ hilb(3)
```

$$\begin{bmatrix} 1.0 & 0.5 & 0.3333333333333333 \\ 0.5 & 0.3333333333333333 & 0.25 \\ 0.3333333333333333 & 0.25 & 0.2 \end{bmatrix} \tag{8}$$

- Note that a vector is a matrix with a single column. The notation $\boldsymbol{v} \in \mathbb{R}^m$, is a customary shorter form of $\boldsymbol{v} \in \mathbb{R}^{m\times 1}$.

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

```
∴ A[:,2]
```

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{9}$$

```
∴ A[:,2]
```

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{10}$$

```
∴ A[:,2]
```

$$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \tag{11}$$

```
∴
```

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \boldsymbol{a}_3\ ]$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
∴ A[2,:]'
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{12}$$

```
∴ A[2,:]'
```

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = [\, a_1 \;\; a_2 \;\; a_3 \,]$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{13}$$

```
∴ A[2,:]'
```

$$[\, -1 \;\; 0 \;\; 1 \,] \tag{14}$$

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

```
∴ A[:,2:3]
```

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{15}$$

```
∴ A[:,2:3]
```

```
∴
```

$$A = \begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$$

```
∴ A=[3 1 2; -1 0 1; 3 4 1]
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 0 & 1 \\ 3 & 4 & 1 \end{bmatrix} \tag{16}$$

```
∴ A[:,2:3]
```

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 4 & 1 \end{bmatrix} \tag{17}$$

```
∴
```

- Single component vector is a scalar $\boldsymbol{v} = [v_1] \equiv v_1$
- Single column vector matrix is a vector $\boldsymbol{A} = [\boldsymbol{a}_1] \equiv \boldsymbol{a}_1$
- Vector addition carries over to matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$

$$\boldsymbol{A} + \boldsymbol{B} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ] + [\ \boldsymbol{b}_1 \ \ \boldsymbol{b}_2 \ \ \dots \ \ \boldsymbol{b}_n\ ]$$

- Vector scaling carries over to matrices

$$\alpha \boldsymbol{A} = \alpha[\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ] = [\ \alpha \boldsymbol{a}_1 \ \ \alpha \boldsymbol{a}_2 \ \ \dots \ \ \alpha \boldsymbol{a}_n\ ]$$

- Identity matrix $\boldsymbol{I} = [\ \boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \dots \ \ \boldsymbol{e}_m\ ] \in \mathbb{R}^{m \times m}$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 + \cdots + x_m \boldsymbol{e}_m = \boldsymbol{I}\boldsymbol{x},\ \boldsymbol{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, \boldsymbol{e}_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

- Matrix transpose in terms of column vectors

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ] \in \mathbb{R}^{m\times n}, \boldsymbol{A}^T = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_n^T \end{bmatrix} \in \mathbb{R}^{n\times m}.$$

- Matrix transpose in terms of components

$$\boldsymbol{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & & a_{m,n} \end{bmatrix} \in \mathbb{R}^{m\times n}, \boldsymbol{A}^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \dots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & & a_{m,n} \end{bmatrix} \in \mathbb{R}^{n\times m}$$

**Definition.** *Consider matrices* $\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \ldots \ \ \boldsymbol{a}_n\ ] \in \mathbb{R}^{m \times n}$, *and* $\boldsymbol{X} = [\ \boldsymbol{x}_1 \ \ \ldots \ \ \boldsymbol{x}_p\ ] \in \mathbb{R}^{n \times p}$. *The matrix product* $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}$ *is a matrix* $\boldsymbol{B} = [\ \boldsymbol{b}_1 \ \ \ldots \ \ \boldsymbol{b}_p\ ] \in \mathbb{R}^{m \times p}$ *with column vectors given by the matrix vector products*

$$\boldsymbol{b}_k = \boldsymbol{A}\boldsymbol{x}_k, \text{for } k = 1, 2 \ldots, p.$$

- A matrix-matrix product is simply a set of matrix-vector products, and hence expresses multiple linear combinations in a concise way.
- The dimensions of the matrices must be compatible, the number of rows of $\boldsymbol{X}$ must equal the number of columns of $\boldsymbol{A}$.
- A matrix-vector product is a special case of a matrix-matrix product when $p = 1$.
- We often write $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{X}$ in terms of columns as

$$[\ \boldsymbol{b}_1 \ \ \ldots \ \ \boldsymbol{b}_p\ ] = \boldsymbol{A}[\ \boldsymbol{x}_1 \ \ \ldots \ \ \boldsymbol{x}_p\ ] = [\ \boldsymbol{A}\boldsymbol{x}_1 \ \ \ldots \ \ \boldsymbol{A}\boldsymbol{x}_p\ ]$$

```
∴ A=[1 0 3; 2 1 4; -1 0 3]
∴ X=[1 -1 0; 1 1 1; 0 1 0]
∴ A*X
∴ [A*X[:,1] A*X[:,2] A*X[:,3]]
∴
```

```
∴ A=[1 0 3; 2 1 4; -1 0 3]
```

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \tag{18}$$

```
∴ X=[1 -1 0; 1 1 1; 0 1 0]
```

```
∴ A*X
```

```
∴ [A*X[:,1] A*X[:,2] A*X[:,3]]
```

```
∴
```

```
∴ A=[1 0 3; 2 1 4; -1 0 3]
```

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \tag{19}$$

```
∴ X=[1 -1 0; 1 1 1; 0 1 0]
```

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

```
∴ A*X
```

```
∴ [A*X[:,1] A*X[:,2] A*X[:,3]]
```

```
∴
```

```
∴ A=[1 0 3; 2 1 4; -1 0 3]
```

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \tag{21}$$

```
∴ X=[1 -1 0; 1 1 1; 0 1 0]
```

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{22}$$

```
∴ A*X
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 3 & 1 \\ -1 & 4 & 0 \end{bmatrix} \tag{23}$$

```
∴ [A*X[:,1] A*X[:,2] A*X[:,3]]
```

```
∴
```

```
∴ A=[1 0 3; 2 1 4; -1 0 3]
```

$$\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ -1 & 0 & 3 \end{bmatrix} \tag{24}$$

```
∴ X=[1 -1 0; 1 1 1; 0 1 0]
```

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{25}$$

```
∴ A*X
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 3 & 1 \\ -1 & 4 & 0 \end{bmatrix} \tag{26}$$

```
∴ [A*X[:,1] A*X[:,2] A*X[:,3]]
```

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 3 & 1 \\ -1 & 4 & 0 \end{bmatrix} \tag{27}$$

```
∴
```

**Definition.** *Consider matrices* $\boldsymbol{A} = [\ a_{i,j}\ ] \in \mathbb{R}^{m \times n}$, *and* $\boldsymbol{X} = [x_{i,j}] \in \mathbb{R}^{n \times p}$. *The matrix product* $\boldsymbol{B} = \boldsymbol{A}\,\boldsymbol{X} = [b_{i,j}]$ *is a matrix* $\boldsymbol{B} \in \mathbb{R}^{m \times p}$ *with components*

$$b_{i,j} = a_{i,1}\,x_{1,j} + a_{i,2}\,x_{2,j} + \cdots + a_{i,n}\,x_{n,j} = \sum_{k=1}^{n} a_{i,k}\,x_{k,j}$$

$$\boldsymbol{B} = [\ \boldsymbol{b}_1\ \ \ldots\ \ \boldsymbol{b}_p\ ], \boldsymbol{b}_1 = \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{m,1} \end{bmatrix} = x_{1,1}\boldsymbol{a}_1 + x_{2,1}\boldsymbol{a}_2 + \cdots + x_{n,1}\boldsymbol{a}_n$$

$$\boldsymbol{B} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,p} \\ b_{2,1} & \cdots & b_{2,p} \\ \vdots & \ddots & \vdots \\ b_{m,1} & & b_{m,p} \end{bmatrix} = \boldsymbol{A}\,\boldsymbol{X} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & & a_{m,n} \end{bmatrix} \begin{bmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & & x_{n,p} \end{bmatrix}$$

$$b_{2,1} = a_{2,1}\,x_{1,1} + a_{2,2}\,x_{2,1} + \cdots + a_{2,n}\,x_{n,1}$$