Overview

- Span of a vector set
- Vector subspaces
- Vector subspace composition
- Vector subspace of a linear mapping and its associated matrix
  - Column space
  - Left null space
- Geometric interpretation of subspaces of Euclidean spaces
- Applications of concept of vector subspace

- Formalize linear combinations by explicit definition of allowed operations ("algebra")

Addition rules for	$\forall \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in V$
$\boldsymbol{a} + \boldsymbol{b} \in V$	Closure
$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$	Associativity
$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$	Commutativity
$\boldsymbol{0} + \boldsymbol{a} = \boldsymbol{a}$	Zero vector
$\boldsymbol{a} + (-\boldsymbol{a}) = \boldsymbol{0}$	Additive inverse
Scaling rules for	$\forall \boldsymbol{a}, \boldsymbol{b} \in V$, $\forall x, y \in S$
$x\boldsymbol{a} \in V$	Closure
$x(\boldsymbol{a} + \boldsymbol{b}) = x\boldsymbol{a} + x\boldsymbol{b}$	Distributivity
$(x + y)\boldsymbol{a} = x\boldsymbol{a} + y\boldsymbol{a}$	Distributivity
$x(y\boldsymbol{a}) = (xy)\boldsymbol{a}$	Composition
$1\boldsymbol{a} = \boldsymbol{a}$	Scalar identity

Table 1. Vector space properties

- Example: $V = \mathbb{R}^m, S = \mathbb{R}$

Definition. *The* span *of vectors* $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, *is the set of vectors reachable by linear combination*

$$\operatorname{span}\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n\} = \{\boldsymbol{b} \in \mathcal{V} : \exists x_1, \ldots, x_n \in \mathcal{S} \text{ such that } \boldsymbol{b} = x_1 \boldsymbol{a}_1 + \ldots x_n \boldsymbol{a}_n\}.$$

The notation used for set on the right hand side is read: ‘‘those vectors $\boldsymbol{b}$ in $\mathcal{V}$ with the property that there exist n scalars $x_1, \ldots, x_n$ to obtain $\boldsymbol{b}$ by linear combination of $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

Example 1. The $\mathbb{R}^2$ plane is described as

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Example 2. Within $\mathbb{R}^2$, the x_1 axis is described as

$$\mathbb{X}_1 = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Definition. (Vector Subspace) . *$\mathcal{U} = (U, S, +, \cdot)$ with $U \neq \varnothing$ is a vector subspace of vector space $\mathcal{V} = (V, S, +, \cdot)$ over the same field of scalars S if $U \subseteq V$ and $\forall a, b \in S$, $\forall \boldsymbol{u}, \boldsymbol{v} \in U$, the linear combination $a\boldsymbol{u} + b\boldsymbol{v} \in U$.*

Definition. *Given two vector subspaces $(U, S, +, \cdot)$, $(V, S, +, \cdot)$ of the space $(W, S, +, \cdot)$, the sum is the set $U + V = \{\boldsymbol{u} + \boldsymbol{v} \colon \boldsymbol{u} \in U, \boldsymbol{v} \in V\}$.*

Definition. *Given two vector subspaces $(U, S, +, \cdot)$, $(V, S, +, \cdot)$ of the space $(W, S, +, \cdot)$, the direct sum is the set $U \oplus V = \{\boldsymbol{u} + \boldsymbol{v} \colon \exists! \boldsymbol{u} \in U, \exists! \boldsymbol{v} \in V\}$. (unique decomposition)*

Definition. *Given two vector subspaces $(U, S, +, \cdot)$, $(V, S, +, \cdot)$ of the space $(W, S, +, \cdot)$, the intersection is the set*

$$\mathcal{U} \cap \mathcal{V} = \{\boldsymbol{x} \colon \boldsymbol{x} \in \mathcal{U}, \boldsymbol{x} \in \mathcal{V}\}.$$

Definition. *Two vector subspaces $\mathcal{U} = (U, S, +, \cdot)$, $\mathcal{V} = (V, S, +, \cdot)$ of the space $\mathcal{W} = (W, S, U +, \cdot)$ are orthogonal subspaces, denoted $\mathcal{U} \perp \mathcal{V}$ if $\boldsymbol{u}^T \boldsymbol{v} = 0$ for any $\boldsymbol{u} \in U, \boldsymbol{v} \in V$.*

Definition. *Two vector subspaces $\mathcal{U} = (U, S, +, \cdot)$, $\mathcal{V} = (V, S, +, \cdot)$ of the space $\mathcal{W} = (W, S, U +, \cdot)$ are orthogonal complements, denoted $\mathcal{U} = \mathcal{V}^\perp$, $\mathcal{V} = \mathcal{U}^\perp$ if they are orthogonal subspaces and $\mathcal{U} \cap \mathcal{V} = \{\boldsymbol{0}\}$, i.e., the null vector is the only common element of both subspaces.*

- Recall that a matrix $\boldsymbol{A}$ can be associated to the linear mapping $\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m$ through

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots \ \ \boldsymbol{a}_n\ ] = [\ \boldsymbol{f}(\boldsymbol{e}_1) \ \ \boldsymbol{f}(\boldsymbol{e}_2) \ \ \ldots \ \ \boldsymbol{f}(\boldsymbol{e}_n)\ ],\ \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$$

Definition. *The column space (or range) of matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the set of vectors reachable by linear combination of the matrix column vectors*

$$C(\boldsymbol{A}) = \mathrm{range}(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m : \exists \boldsymbol{x} \in \mathbb{R}^n \,\text{such that}\, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m.$$

Definition. *The left null space of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the set*

$$N(\boldsymbol{A}^T) = \mathrm{null}(\boldsymbol{A}^T) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n.$$

Definition. *The row space (or corange) of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the set*

$$R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \mathrm{range}(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n : \exists \boldsymbol{y} \in \mathbb{R}^m \, \boldsymbol{c} = \boldsymbol{A}^T\boldsymbol{y}\} \subseteq \mathbb{R}^n$$

Definition. *The null space of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the set*

$$N(\boldsymbol{A}) = \mathrm{null}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \subseteq \mathbb{R}^n$$

- The $\mathbb{R}^2$ plane
$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$
- A line in the $\mathbb{R}^2$ plane
$$L_{p,q} = \left\{ \begin{bmatrix} ap \\ aq \end{bmatrix} : a \in \mathbb{R} \right\}.$$
- The x_1, x_2 axes
$$L_{1,0} = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}, L_{0,1} = \left\{ \begin{bmatrix} 0 \\ a \end{bmatrix} : a \in \mathbb{R} \right\},$$
- $\mathbb{R}^3$, three-dimensional space
$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R} \right\}.$$
- Lines in $\mathbb{R}^3$
$$L_{p,q,r} = \left\{ \begin{bmatrix} ap \\ aq \\ ar \end{bmatrix} : a \in \mathbb{R} \right\}$$
- Planes in $\mathbb{R}^3$
$$P_{\boldsymbol{n}} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : \boldsymbol{n}^T \boldsymbol{x} = n_1 x_1 + n_2 x_2 + n_3 x_3 = 0, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

- Define a function to give a spanning set for the column space

```
∴ function colspace(A,p=6)
    return round.(Matrix(qr(A).Q)[:,1:rank(A)],digits=p)
  end;
```

```
∴ short(x) = round(x,digits=6);
```

```
∴ short(pi)
```

```
∴ colspace([1; 0; 0])
```

```
∴
```

- Julia already has a function to give a spanning set for the null set

```
∴ nullspace([1 0 0])
```

- Define a function to give a spanning set for the column space

```
∴ function colspace(A,p=6)
    return round.(Matrix(qr(A).Q)[:,1:rank(A)],digits=p)
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```
∴ short(x) = round(x,digits=6);
```

```
∴ short(pi)
```

3.141593

```
∴ colspace([1; 0; 0])
```

```
∴
```

- Julia already has a function to give a spanning set for the null set

```
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- Define a function to give a spanning set for the column space

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```

```
∴ short(x) = round(x,digits=6);
```

```
∴ short(pi)
```

3.141593

```
∴ colspace([1; 0; 0])
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \tag{1}$$

```
∴
```

- Julia already has a function to give a spanning set for the null set

```
∴ nullspace([1 0 0])
```

- Define a function to give a spanning set for the column space

```
∴ function colspace(A,p=6)
    return round.(Matrix(qr(A).Q)[:,1:rank(A)],digits=p)
  end;
```

```
∴ short(x) = round(x,digits=6);
```

```
∴ short(pi)
```

3.141593

```
∴ colspace([1; 0; 0])
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \tag{2}$$

```
∴
```

- Julia already has a function to give a spanning set for the null set

```
∴ nullspace([1 0 0])
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{3}$$

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane

```
∴ A=[1; 0; 0]; colspace(A)
∴ nullspace(A')
∴ [colspace(A) nullspace(A')]
∴
```

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane

```
∴ A=[1; 0; 0]; colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \quad (4)$$

```
∴ nullspace(A')
```

```
∴ [colspace(A) nullspace(A')]
```

```
∴
```

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane

```
∴ A=[1; 0; 0]; colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \quad (5)$$

```
∴ nullspace(A')
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad (6)$$

```
∴ [colspace(A) nullspace(A')]
```

```
∴
```

Example 1

$$A = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane

```
∴ A=[1; 0; 0]; colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \tag{7}$$

```
∴ nullspace(A')
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{8}$$

```
∴ [colspace(A) nullspace(A')]
```

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{9}$$

```
∴
```

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2\ ],\ \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

The columns of $\boldsymbol{A}$ are colinear, $\boldsymbol{a}_2 = -\boldsymbol{a}_1$, and the column space $C(\boldsymbol{A})$ is the y_1-axis, and the left null space $N(\boldsymbol{A}^T)$ is the y_2y_3-plane, as before.

```
∴ A=[1 -1; 0 0; 0 0]; CA=colspace(A)
∴ NAt=short.(nullspace(A'))
∴ [CA NAt]
∴
```

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [\ a_1 \ \ a_2\ ],\ A^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

The columns of A are colinear, $a_2 = -a_1$, and the column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane, as before.

```
∴ A=[1 -1; 0 0; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \tag{10}$$

```
∴ NAt=short.(nullspace(A'))
```

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [\ a_1 \ \ a_2\ ],\ A^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

The columns of A are colinear, $a_2 = -a_1$, and the column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane, as before.

```
∴ A=[1 -1; 0 0; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \quad (11)$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \quad (12)$$

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = [\, a_1 \;\; a_2 \,],\; A^T = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

The columns of A are colinear, $a_2 = -a_1$, and the column space $C(A)$ is the y_1-axis, and the left null space $N(A^T)$ is the y_2y_3-plane, as before.

```
∴ A=[1 -1; 0 0; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 \\ 0.0 \\ 0.0 \end{bmatrix} \tag{13}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 & 0.0 \\ 1.0 & 0.0 \\ 0.0 & 1.0 \end{bmatrix} \tag{14}$$

```
∴ [CA NAt]
```

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{15}$$

```
∴
```

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1y_2-plane, and the left null space $N(A^T)$ is the y_3-axis.

```
∴ A=[1 0; 0 1; 0 0]; CA=colspace(A)
∴ NAt=short.(nullspace(A'))
∴ [CA NAt]
∴
```

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

The column space $C(\boldsymbol{A})$ is the y_1y_2-plane, and the left null space $N(\boldsymbol{A}^T)$ is the y_3-axis.

```
∴ A=[1 0; 0 1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} \tag{16}$$

```
∴ NAt=short.(nullspace(A'))
```

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

The column space $C(A)$ is the y_1y_2-plane, and the left null space $N(A^T)$ is the y_3-axis.

```
∴ A=[1 0; 0 1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} \tag{17}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \tag{18}$$

```
∴ [CA NAt]
```

```
∴
```

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

The column space $C(\boldsymbol{A})$ is the y_1y_2-plane, and the left null space $N(\boldsymbol{A}^T)$ is the y_3-axis.

```
∴ A=[1 0; 0 1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \\ 0.0 & 0.0 \end{bmatrix} \tag{19}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \tag{20}$$

```
∴ [CA NAt]
```

$$\begin{bmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{21}$$

```
∴
```

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \boldsymbol{A}^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

the same $C(\boldsymbol{A})$, $N(\boldsymbol{A}^T)$ are obtained, albeit with a different set of spanning vectors returned by `colspace`.

```
∴ A=[1 1; 1 -1; 0 0]; CA=colspace(A)
∴ NAt=short.(nullspace(A'))
∴ [CA NAt]
∴
```

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

the same $C(A)$, $N(A^T)$ are obtained, albeit with a different set of spanning vectors returned by `colspace`.

```
∴ A=[1 1; 1 -1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \\ 0.0 & 0.0 \end{bmatrix} \tag{22}$$

```
∴ NAt=short.(nullspace(A'))
```

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, \; A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

the same $C(A)$, $N(A^T)$ are obtained, albeit with a different set of spanning vectors returned by `colspace`.

```
∴ A=[1 1; 1 -1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \\ 0.0 & 0.0 \end{bmatrix} \tag{23}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \tag{24}$$

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix},$$

the same $C(A)$, $N(A^T)$ are obtained, albeit with a different set of spanning vectors returned by `colspace`.

```
∴ A=[1 1; 1 -1; 0 0]; CA=colspace(A)
```

$$\begin{bmatrix} -0.707107 & -0.707107 \\ -0.707107 & 0.707107 \\ 0.0 & 0.0 \end{bmatrix} \tag{25}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \end{bmatrix} \tag{26}$$

```
∴ [CA NAt]
```

$$\begin{bmatrix} -0.707107 & -0.707107 & 0.0 \\ -0.707107 & 0.707107 & 0.0 \\ 0.0 & 0.0 & 1.0 \end{bmatrix} \tag{27}$$

```
∴
```

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3\ ],\ A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{bmatrix},\ A^T\boldsymbol{y} = \begin{bmatrix} \boldsymbol{a}_1^T\boldsymbol{y} \\ \boldsymbol{a}_2^T\boldsymbol{y} \\ \boldsymbol{a}_3^T\boldsymbol{y} \end{bmatrix}.$$

$\boldsymbol{a}_3 = \boldsymbol{a}_1 + 2\boldsymbol{a}_2 \Rightarrow$ $A^T\boldsymbol{y} = \boldsymbol{0}$ is satisfied by vectors of form $\boldsymbol{y} = [\ a \ \ 0 \ \ -a\ ]$, $\forall a \in \mathbb{R}$.

```
∴ A=[1 1 3; 1 -1 -1; 1 1 3]; CA=colspace(A)
∴ NAt=short.(nullspace(A'))
∴ [CA NAt]
∴
```

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \boldsymbol{a}_3\ ], A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \boldsymbol{a}_3^T \end{bmatrix}, A^T\boldsymbol{y} = \begin{bmatrix} \boldsymbol{a}_1^T\boldsymbol{y} \\ \boldsymbol{a}_2^T\boldsymbol{y} \\ \boldsymbol{a}_3^T\boldsymbol{y} \end{bmatrix}.$$

$\boldsymbol{a}_3 = \boldsymbol{a}_1 + 2\boldsymbol{a}_2 \Rightarrow \quad A^T\boldsymbol{y} = \boldsymbol{0}$ is satisfied by vectors of form $\boldsymbol{y} = [\ a \ \ 0 \ \ -a\ ]$, $\forall a \in \mathbb{R}$.

```
∴ A=[1 1 3; 1 -1 -1; 1 1 3]; CA=colspace(A)
```

$$\begin{bmatrix} -0.5773502691896257 & 0.40824829046386313 \\ -0.5773502691896257 & -0.816496580927726 \\ -0.5773502691896257 & 0.40824829046386313 \end{bmatrix} \tag{28}$$

```
∴ NAt=short.(nullspace(A'))
```

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}, A^T y = \begin{bmatrix} a_1^T y \\ a_2^T y \\ a_3^T y \end{bmatrix}.$$

$a_3 = a_1 + 2a_2 \Rightarrow \quad A^T y = 0$ is satisfied by vectors of form $y = \begin{bmatrix} a & 0 & -a \end{bmatrix}, \forall a \in \mathbb{R}$.

```
∴ A=[1 1 3; 1 -1 -1; 1 1 3]; CA=colspace(A)
```

$$\begin{bmatrix} -0.5773502691896257 & 0.40824829046386313 \\ -0.5773502691896257 & -0.816496580927726 \\ -0.5773502691896257 & 0.40824829046386313 \end{bmatrix} \tag{29}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} -0.707107 \\ 0.0 \\ 0.707107 \end{bmatrix} \tag{30}$$

```
∴ [CA NAt]
```

```
∴
```

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -1 & -1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & -1 & 3 \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}, A^T y = \begin{bmatrix} a_1^T y \\ a_2^T y \\ a_3^T y \end{bmatrix}.$$

$a_3 = a_1 + 2a_2 \Rightarrow$ $A^T y = 0$ is satisfied by vectors of form $y = \begin{bmatrix} a & 0 & -a \end{bmatrix}, \forall a \in \mathbb{R}$.

```
∴ A=[1 1 3; 1 -1 -1; 1 1 3]; CA=colspace(A)
```

$$\begin{bmatrix} -0.5773502691896257 & 0.408248290463863 13 \\ -0.5773502691896257 & -0.816496580927726 \\ -0.5773502691896257 & 0.40824829046386313 \end{bmatrix} \tag{31}$$

```
∴ NAt=short.(nullspace(A'))
```

$$\begin{bmatrix} -0.707107 \\ 0.0 \\ 0.707107 \end{bmatrix} \tag{32}$$

```
∴ [CA NAt]
```

$$\begin{bmatrix} -0.5773502691896257 & 0.40824829046386313 & -0.707107 \\ -0.5773502691896257 & -0.816496580927726 & 0.0 \\ -0.5773502691896257 & 0.40824829046386313 & 0.707107 \end{bmatrix} \tag{33}$$

```
∴
```