Overview

- Linear dependence and independence
- Orthogonal, orthonormal vector sets
- Orthogonal matrices
- Basis, dimension
- Realistic application of vector operations framework: ECG representation and compression
  - Sampling
  - Recursive definition of $\boldsymbol{I}$
  - Hadamard-Walsh matrices
  - Compression by truncation of linear combinations.

- Let $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ be a matrix with n column vectors, each with m components

$$\boldsymbol{A} = [\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \ \ \boldsymbol{a}_n\ ], \boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n \in \mathbb{R}^m$$

- $\boldsymbol{A}$ can be thought of as representing a linear mapping $\boldsymbol{f}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$, $\mathbb{R}^n \xrightarrow{\boldsymbol{A}} \mathbb{R}^m$

$$\boldsymbol{f}: \mathbb{R}^n \to \mathbb{R}^m, \boldsymbol{A} = [\ \boldsymbol{f}(\boldsymbol{e}_1) \ \ \boldsymbol{f}(\boldsymbol{e}_2) \ \ \dots \ \ \boldsymbol{f}(\boldsymbol{e}_n)\ ], \boldsymbol{I}_n \in \mathbb{R}^{n \times n}, \boldsymbol{I}_n = [\ \boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \dots \ \ \boldsymbol{e}_n\ ]$$

- *Column space*, $C(\boldsymbol{A}) = \{\boldsymbol{b} \in \mathbb{R}^m | \ \exists \boldsymbol{x} \in \mathbb{R}^n \text{ such that } \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *reachable* by linear combination of columns of $\boldsymbol{A}$
- *Left null space*, $N(\boldsymbol{A}^T) = \{\boldsymbol{y} \in \mathbb{R}^m | \ \boldsymbol{A}^T \boldsymbol{y} = 0\} \subseteq \mathbb{R}^m$, the part of $\mathbb{R}^m$ *not reachable* by linear combination of columns of $\boldsymbol{A}$
- *Row space*, $R(\boldsymbol{A}) = C(\boldsymbol{A}^T) = \{\boldsymbol{c} \in \mathbb{R}^n | \ \exists \boldsymbol{y} \in \mathbb{R}^m \text{ such that } \boldsymbol{c} = \boldsymbol{A}^T \boldsymbol{y}\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^n$ *reachable* by linear combination of rows of $\boldsymbol{A}$
- *Null space*, $N(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n | \ \boldsymbol{A}\boldsymbol{x} = 0\} \subseteq \mathbb{R}^n$, the part of $\mathbb{R}^n$ *not reachable* by linear combination of rows of $\boldsymbol{A}$

- Zero product property of scalar multiplication: $ax = 0 \Rightarrow a = 0$ or $x = 0$
- Matrix-vector counterexamples of zero product property
  - $\boldsymbol{A} = [\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ ] = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \boldsymbol{0}$
  - $\boldsymbol{B} = [\ \boldsymbol{b}_1\ \ \boldsymbol{b}_2\ \ \boldsymbol{b}_3\ ] = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 4 \\ 3 & 1 & 7 \end{bmatrix}, \boldsymbol{B}\boldsymbol{x} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 4 \\ 3 & 1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{0}$
- Matrix-vector example satisfying the zero product property $\boldsymbol{I}\boldsymbol{x} = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$
- Question: how to distinguish between above examples?
- Note:
  - $\boldsymbol{a}_1 = \boldsymbol{a}_2$
  - $\boldsymbol{b}_3 = 2\boldsymbol{b}_1 + \boldsymbol{b}_2$

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are* linearly dependent *if there exist n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, at least one of which is different from zero such that*

$$x_1\boldsymbol{a}_1 + \ldots x_n\boldsymbol{a}_n = \mathbf{0}$$

Note that $\{\mathbf{0}\}$, with $\mathbf{0} \in \mathcal{V}$ is a linearly dependent set of vectors since $1 \cdot \mathbf{0} = \mathbf{0}$.

The converse of linear dependence is linear independence, a member of the set cannot be expressed as a non-trivial linear combination of the other vectors

Definition. *The vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n \in \mathcal{V}$, are* linearly independent *if the* only *n scalars, $x_1, \ldots, x_n \in \mathcal{S}$, that satisfy*

$$x_1\boldsymbol{a}_1 + \ldots x_n\boldsymbol{a}_n = \mathbf{0}, \tag{1}$$

are $x_1 = 0$, $x_2 = 0$,...,$x_n = 0$.

The choice $\boldsymbol{x} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix}^T = \mathbf{0}$ that always satisfies (1) is called a *trivial solution*. We can restate linear independence as (1) being satisfied *only* by the trivial solution.

Recall:

Definition. *The null space of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *is the set*

$$N(\boldsymbol{A}) = \text{null}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\} \leq \mathbb{R}^n \tag{2}$$

- If $N(\boldsymbol{A}) = \{\boldsymbol{0}\}$ then the column vectors of $\boldsymbol{A}$ are linearly independent, since the only way to satisfy (1) is by the trivial solution $\boldsymbol{x} = \boldsymbol{0}$

Definition. *The left null space of a matrix* $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ *is the set*

$$N(\boldsymbol{A}^T) = \text{null}(\boldsymbol{A}^T) = \{\boldsymbol{y} \in \mathbb{R}^m | \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}\} \leq \mathbb{R}^n \tag{3}$$

- If $N(\boldsymbol{A}^T) = \{\boldsymbol{0}\}$ then the row vectors of $\boldsymbol{A}$ are linearly independent, since the only way to satisfy (1) is by the trivial solution $\boldsymbol{x} = \boldsymbol{0}$

Definition. *The column vectors* $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n \in \mathbb{R}^m$ *of matrix* $\boldsymbol{U} \in \mathbb{R}^{m \times n}$ *are orthogonal if*

$$\boldsymbol{U}^T\boldsymbol{U} = \operatorname{diag}(\|\boldsymbol{u}_1\|^2, \ldots, \|\boldsymbol{u}_n\|^2).$$

Definition. *The column vectors* $\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n \in \mathbb{R}^m$ *of matrix* $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$ *are orthonormal if*

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}.$$

Definition. *The matrix* $\boldsymbol{Q} \in \mathbb{R}^{m \times m}$ *is orthogonal if*

$$\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}.$$

Example. The reflection matrix across direction $\boldsymbol{q}$, $\|\boldsymbol{q}\| = 1$ in $\mathbb{R}^m$,$\boldsymbol{R}_{\boldsymbol{q}} = 2\,\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I}$, is orthogonal

$$\boldsymbol{R}_{\boldsymbol{q}}\boldsymbol{R}_{\boldsymbol{q}}^T = (2\,\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I})(2\,\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I})^T = (2\,\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I})(2\,\boldsymbol{q}\boldsymbol{q}^T - \boldsymbol{I}) = 4\boldsymbol{q}\boldsymbol{q}^T\boldsymbol{q}\boldsymbol{q}^T - 4\boldsymbol{q}\boldsymbol{q}^T - I = \boldsymbol{I}$$

since $\boldsymbol{q}\boldsymbol{q}^T\boldsymbol{q}\boldsymbol{q}^T = \boldsymbol{q}\,(\boldsymbol{q}^T\boldsymbol{q})\,\boldsymbol{q}^T = \boldsymbol{q}\,(1)\,\boldsymbol{q}^T = \boldsymbol{q}\,\boldsymbol{q}^T$.

Suppose in $\mathcal{V} = (V, \mathbb{R}, +, \cdot)$ the set $\mathcal{B} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots\}$ spans V, $V = \operatorname{span} \mathcal{B}$. Adding another vector does not change the span $\operatorname{span} \mathcal{B} = \operatorname{span} (\mathcal{B} \cup \{\boldsymbol{b}\})$. Intuitively $\mathcal{B} \cup \{\boldsymbol{b}\}$ contains a redundant vector, it is not a minimal spanning set. Avoid redundancy by defining minimal spanning sets.

Definition. *A set of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in V$ is a basis for vector space $\mathcal{V}$ if:*

1. $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ *are linearly independent;*
2. $\operatorname{span}\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} = V$.

Adding another vector $\boldsymbol{b} \in$ leads to a linearly dependent set $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n, \boldsymbol{b}\}$.

Definition. *The number of vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n \in V$ within a basis is the dimension of the vector space $\mathcal{V}$.*

- $C(\boldsymbol{A})$ the column space of $\boldsymbol{A}$, $C(\boldsymbol{A}) \le \mathbb{R}^m$
- $C(\boldsymbol{A}^T)$ the row space of $\boldsymbol{A}$, $C(\boldsymbol{A}^T) \le \mathbb{R}^n$
- $N(\boldsymbol{A})$ the null space of $\boldsymbol{A}$, $N(\boldsymbol{A}) \le \mathbb{R}^n$
- $N(\boldsymbol{A}^T)$ the left null space of $\boldsymbol{A}$, or null space of $\boldsymbol{A}^T$, $N(\boldsymbol{A}^T) \le \mathbb{R}^m$.

The dimensions of these subspaces arise so often in applications to warrant formal definition.

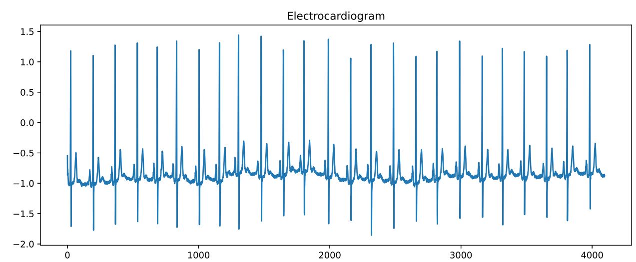
Definition. *The rank of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the dimension of its column space.*

Definition. *The nullity of a matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ is the dimension of its null space.*

Dimension of column space equals dimension of row space

$$\boldsymbol{b} = \boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{a}_1 + \cdots + x_n \boldsymbol{a}_n \Leftrightarrow \boldsymbol{b}^T = (\boldsymbol{A}\boldsymbol{x})^T = \boldsymbol{x}^T \boldsymbol{A}^T = x_1 \boldsymbol{a}_1^T + \cdots + x_n \boldsymbol{a}_n^T .$$

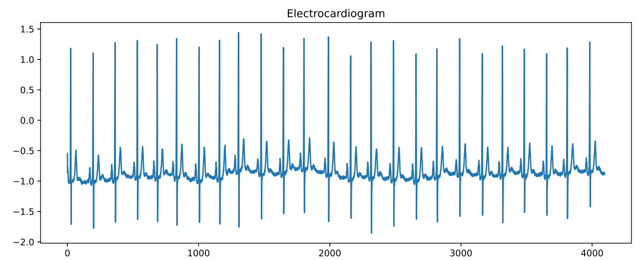
```
∴ using MAT
∴ DataFileName = homedir()*"/courses/MATH347DS/data/ecg/ECGData.mat";
∴ DataFile = matopen(DataFileName,"r");
∴ dict = read(DataFile,"ECGData");
∴ data = dict["Data"]';
∴ size(data)
∴ q=12; m=2^q; k=15; b=data[1:m,k];
∴ figure(1); clf(); plot(b); title("Electrocardiogram");
∴ cd(homedir()*"/courses/MATH347DS/images"); savefig("S04Fig01.eps");
∴
```



```
∴ using MAT
∴ DataFileName = homedir()*"/courses/MATH347DS/data/ecg/ECGData.mat";
∴ DataFile = matopen(DataFileName,"r");
∴ dict = read(DataFile,"ECGData");
∴ data = dict["Data"]';
∴ size(data)
```

$$\begin{bmatrix} 65536 \\ 162 \end{bmatrix} \tag{4}$$

```
∴ q=12; m=2^q; k=15; b=data[1:m,k];
∴ figure(1); clf(); plot(b); title("Electrocardiogram");
∴ cd(homedir()*"/courses/MATH347DS/images"); savefig("S04Fig01.eps");
∴
```



- Consider $m = 2^q$, denote $\boldsymbol{I}_q$ the identity matrix of size $m \times m = 2^q \times 2^q$

$$\boldsymbol{I}_0 = [1], \boldsymbol{I}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_1 \end{bmatrix} = \begin{bmatrix} 1 \cdot \boldsymbol{I}_1 & 0 \cdot \boldsymbol{I}_1 \\ 0 \cdot \boldsymbol{I}_1 & 1 \cdot \boldsymbol{I}_1 \end{bmatrix}, \boldsymbol{I}_3 = \begin{bmatrix} 1 \cdot \boldsymbol{I}_2 & 0 \cdot \boldsymbol{I}_2 \\ 0 \cdot \boldsymbol{I}_2 & 1 \cdot \boldsymbol{I}_2 \end{bmatrix}.$$

Definition. *The exterior product of matrices* $\boldsymbol{A} = [a_{ij}] \in \mathbb{R}^{m \times n}$ *and* $\boldsymbol{B} \in \mathbb{R}^{p \times q}$ *is the matrix* $\boldsymbol{C} \in \mathbb{R}^{(mp) \times (nq)}$

$$\boldsymbol{C} = \boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & a_{12}\boldsymbol{B} & \ldots & a_{1n}\boldsymbol{B} \\ a_{21}\boldsymbol{B} & a_{21}\boldsymbol{B} & \ldots & a_{2n}\boldsymbol{B} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}\boldsymbol{B} & a_{m2}\boldsymbol{B} & \ldots & a_{mn}\boldsymbol{B} \end{bmatrix}.$$

- $\boldsymbol{I}_2 = \boldsymbol{I}_1 \otimes \boldsymbol{I}_1, \boldsymbol{I}_3 = \boldsymbol{I}_1 \otimes \boldsymbol{I}_2$

- Choose a differnt set of starting matrices and obtain another sequence $\boldsymbol{H}_q \in \mathbb{R}^{m \times m}$, $m = 2^q$

$$\boldsymbol{H}_0 = [1], \boldsymbol{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \boldsymbol{H}_q = \boldsymbol{H}_1 \otimes \boldsymbol{H}_{q-1}.$$

```
∴ using Hadamard
```

```
∴ H2=hadamard(2^2)
```

```
∴ H2=hadamard(2^3)
```

- Choose a differnt set of starting matrices and obtain another sequence $\boldsymbol{H}_q \in \mathbb{R}^{m\times m}$, $m = 2^q$

$$\boldsymbol{H}_0 = [1], \boldsymbol{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \boldsymbol{H}_q = \boldsymbol{H}_1 \otimes \boldsymbol{H}_{q-1}.$$

```
∴ using Hadamard
```

```
∴ H2=hadamard(2^2)
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{5}$$

```
∴ H2=hadamard(2^3)
```

- Choose a differnt set of starting matrices and obtain another sequence $\boldsymbol{H}_q \in \mathbb{R}^{m\times m}$, $m = 2^q$

$$\boldsymbol{H}_0 = [1], \boldsymbol{H}_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \boldsymbol{H}_q = \boldsymbol{H}_1 \otimes \boldsymbol{H}_{q-1}.$$

```
∴ using Hadamard
```

```
∴ H2=hadamard(2^2)
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{6}$$

```
∴ H2=hadamard(2^3)
```

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \tag{7}$$

- The pattern of components in $\boldsymbol{H}_q$ less discernable than that in $\boldsymbol{I}_q$
- Visualize the non-zero elements in matrices

```
∴ q=5; m=2^q; Iq=Matrix(1.0I,m,m); Hq=hadamard(m);
∴ clf(); subplot(1,2,1); spy(Iq); subplot(1,2,2); spy(Hq.+1);
∴ cd(homedir()*"/courses/MATH347DS/images"); savefig("S03Fig02.eps");
∴
```

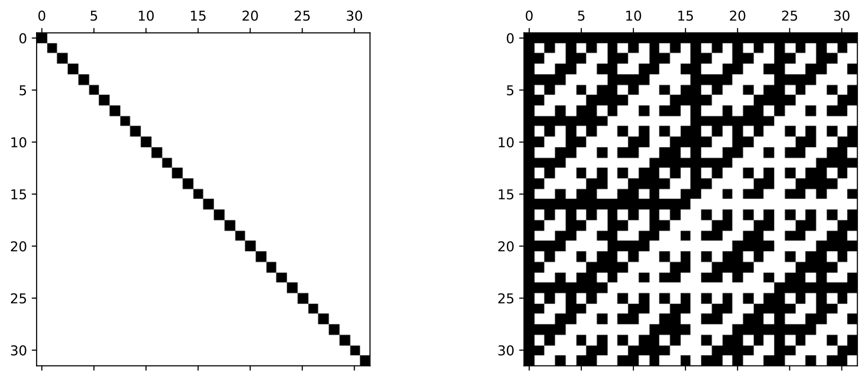


Figure 1. Structures of $\boldsymbol{I}_5$, $\boldsymbol{H}_5$

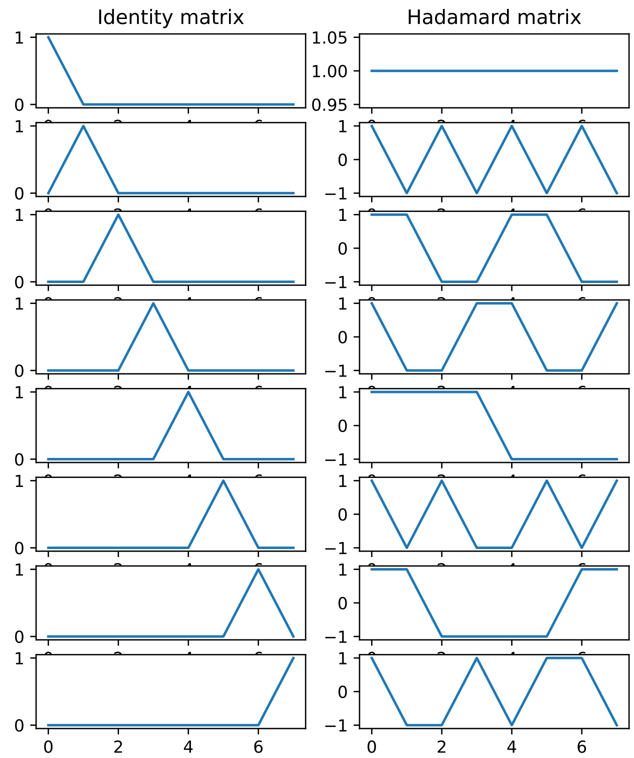


Figure 2. Comparison of column vectors of $\boldsymbol{I}, \boldsymbol{H}$.

Vectors of $\boldsymbol{I}$ sample one moment in time, vectors of $\boldsymbol{H}$ sample multiple moments

- Idea: drop some of the terms in the linear combinations. Instead of

$$\boldsymbol{b} = \boldsymbol{I}\boldsymbol{b} = b_1\,\boldsymbol{e}_1 + \cdots + b_m\,\boldsymbol{e}_m = c_1\,\boldsymbol{h}_1 + \cdots + c_m\,\boldsymbol{h}_m = \boldsymbol{H}\boldsymbol{c}$$

  define

$$\boldsymbol{u} = b_1\,\boldsymbol{e}_1 + \cdots + b_n\,\boldsymbol{e}_n, \boldsymbol{v} = c_1\,\boldsymbol{h}_1 + \cdots + c_n\,\boldsymbol{h}_n$$

```
∴ q=12; m=2^q; k=15; b=data[1:m,k];
∴ Iq=Matrix(1.0I,m,m); Hq=hadamard(m); c=(1/m)*transpose(Hq)*b;
∴ n=2^10; u=Iq[:,1:n]*b[1:n]; v=Hq[:,1:n]*c[1:n];
∴ figure(2); clf(); subplot(3,1,1); plot(b);
∴ subplot(3,1,2); plot(u);
∴ subplot(3,1,3); plot(v);
∴ cd(homedir()*"/courses/MATH347DS/images"); savefig("S04Fig03.eps");
∴
```

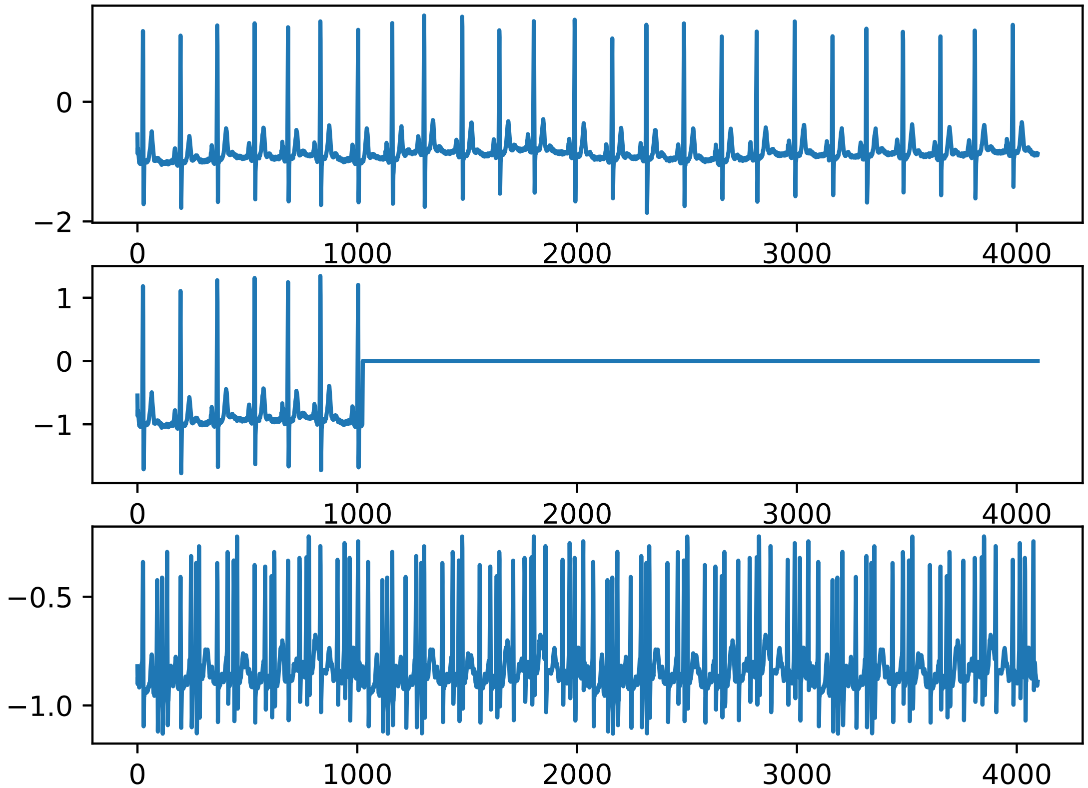


Figure 3. Top: original ECG, Middle: Truncation in $\boldsymbol{I}$-basis, Bottom: Truncation in $\boldsymbol{H}$-basis