Lemma 1. *Let $\mathcal{U}, \mathcal{V}$, be subspaces of vector space $\mathcal{W}$. Then $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ if and only if*

i. $\mathcal{W} = \mathcal{U} + \mathcal{V}$, *and*

ii. $\mathcal{U} \cap \mathcal{V} = \{\mathbf{0}\}$.

Proof. $\mathcal{W} = \mathcal{U} \oplus \mathcal{V} \Rightarrow \mathcal{W} = \mathcal{U} + \mathcal{V}$ *by definition of direct sum, sum of vector subspaces. To prove that* $\mathcal{W} = \mathcal{U} \oplus \mathcal{V} \Rightarrow \mathcal{U} \cap \mathcal{V} = \{\mathbf{0}\}$, *consider* $\boldsymbol{w} \in \mathcal{U} \cap \mathcal{V}$. *Since* $\boldsymbol{w} \in \mathcal{U}$ *and* $\boldsymbol{w} \in \mathcal{V}$ *write*

$$\boldsymbol{w} = \boldsymbol{w} + \mathbf{0} \quad (\boldsymbol{w} \in \mathcal{U}, \mathbf{0} \in \mathcal{V}), \quad \boldsymbol{w} = \mathbf{0} + \boldsymbol{w} \quad (\mathbf{0} \in \mathcal{U}, \boldsymbol{w} \in \mathcal{V}),$$

and since expression $\boldsymbol{w} = \boldsymbol{u} + \boldsymbol{v}$ *is unique, it results that* $\boldsymbol{w} = \mathbf{0}$. *Now assume (i),(ii) and establish an unique decomposition. Assume there might be two decompositions of* $\boldsymbol{w} \in \mathcal{W}$, $\boldsymbol{w} = \boldsymbol{u}_1 + \boldsymbol{v}_1$, $\boldsymbol{w} = \boldsymbol{u}_2 + \boldsymbol{v}_2$, *with* $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{U}$, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathcal{V}$. *Obtain* $\boldsymbol{u}_1 + \boldsymbol{v}_1 = \boldsymbol{u}_2 + \boldsymbol{v}_2$, *or* $\boldsymbol{x} = \boldsymbol{u}_1 - \boldsymbol{u}_2 = \boldsymbol{v}_2 - \boldsymbol{v}_1$. *Since* $\boldsymbol{x} \in \mathcal{U}$ *and* $\boldsymbol{x} \in \mathcal{V}$ *it results that* $\boldsymbol{x} = \mathbf{0}$, *and* $\boldsymbol{u}_1 = \boldsymbol{u}_2$, $\boldsymbol{v}_1 = \boldsymbol{v}_2$, *i.e., the decomposition is unique.* $\square$