Overview

- Orthogonal projectors
- Gaussian elimination
- Row echelon reduction
- Matrix rank from row echelon reduction
- LU-factorization

- Consider the linear system $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ with $\boldsymbol{A}\in\mathbb{R}^{m\times n}$, $\boldsymbol{x}\in\mathbb{R}^n$, $\boldsymbol{b}\in\mathbb{R}^m$. Orthogonal projectors and knowledge of the four fundamental matrix subspaces allows us to succintly express whether there exist no solutions, a single solution of an infinite number of solutions:
  - Consider the factorization $\boldsymbol{Q}\boldsymbol{R}=\boldsymbol{A}$, the orthogonal projector $\boldsymbol{P}=\boldsymbol{Q}\boldsymbol{Q}^T$, and the complementary orthogonal projector $\boldsymbol{I}-\boldsymbol{P}$
  - If $\|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{b}\|\neq 0$, then $\boldsymbol{b}$ has a component outside the column space of $\boldsymbol{A}$, and $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$ has no solution
  - If $\|(\boldsymbol{I}-\boldsymbol{P})\boldsymbol{b}\|=0$, then $\boldsymbol{b}\in C(\boldsymbol{Q})=C(\boldsymbol{A})$ and the system has at least one solution
  - If $N(\boldsymbol{A})=\{\boldsymbol{0}\}$ (null space only contains the zero vector, i.e., null space of dimension 0) the system has a unique solution
  - If $\dim N(\boldsymbol{A})=n-r>0$, then a vector $\boldsymbol{y}\in N(\boldsymbol{A})$ in the null space is written as

$$\boldsymbol{y}=c_1\boldsymbol{z}_1+\ldots+c_{n-r}\boldsymbol{z}_{n-r}$$

    and if $\boldsymbol{x}$ is a solution of $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}$, so is $\boldsymbol{x}+\boldsymbol{y}$, since

$$\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{y})=\boldsymbol{A}\boldsymbol{x}+c_1\boldsymbol{A}\boldsymbol{z}_1+\ldots+c_{n-r}\boldsymbol{A}\boldsymbol{z}_{n-r}=\boldsymbol{b}+\boldsymbol{0}+\ldots+\boldsymbol{0}=\boldsymbol{b}$$

    The linear system has an $(n-r)$-parameter family of solutions

- Idea: make one fewer unknown appear in each equation. Use first equation to eliminate x_1 in equations 2,3

$$\begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 - x_2 - x_3 &= 1 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ -5x_2 + 3x_3 &= -2 \\ -7x_2 + 2x_3 &= -5 \end{cases}$$

- Use second equation to eliminate x_2 in equation 3

$$\begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ -5x_2 + 3x_3 &= -2 \\ -7x_2 + 2x_3 &= -5 \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 &= 2 \\ -5x_2 + 3x_3 &= -2 \\ -\frac{11}{5}x_3 &= -\frac{11}{5} \end{cases}$$

- Start finding components from last to first to obtain $x_3 = 1$, $x_2 = 1$,$x_1 = 1$

- Explicitly writing the unknowns x_1, x_2, x_3 is not necessary. Intoduce the "bordered" matrix

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix}$$

- Define allowed operations:
  - multiply a row by a non-zero scalar
  - add a row to another
- Bordered matrices obtained by the allowed operations are said to be *similar*, in that the solution of the linear system stays the same

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix}$$

- To find solution, use allowed operations to make an identity matrix appear

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- The above constitute "Gaussian elimination"

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix}$$

```
∴ A=[1. 2 -1 2; 2 -1 1 2; 3 -1 -1 1]; A[2,:]=A[2,:]-2*A[1,:]; A[3,:]=A[3,:]-3*A[1,:];
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∴ A
```

```
∴ A[3,:]=A[3,:]-(7/5)*A[2,:]; A
```

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$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

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```

```
∴ A
```

$$\begin{bmatrix} 1.0 & 2.0 & -1.0 & 2.0 \\ 0.0 & -5.0 & 3.0 & -2.0 \\ 0.0 & -7.0 & 2.0 & -5.0 \end{bmatrix} \quad (1)$$

```
∴ A[3,:]=A[3,:]-(7/5)*A[2,:]; A
```

- To find solution, use allowed operations to make an identity matrix appear

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & -\frac{11}{5} & -\frac{11}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- The above constitute "Gaussian elimination"

$$\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & -1 & 1 & 2 \\ 3 & -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 3 & -2 \\ 0 & -7 & 2 & -5 \end{bmatrix}$$

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```

```
∴ A
```

$$\begin{bmatrix} 1.0 & 2.0 & -1.0 & 2.0 \\ 0.0 & -5.0 & 3.0 & -2.0 \\ 0.0 & -7.0 & 2.0 & -5.0 \end{bmatrix} \tag{2}$$

```
∴ A[3,:]=A[3,:]-(7/5)*A[2,:]; A
```

$$\begin{bmatrix} 1.0 & 2.0 & -1.0 & 2.0 \\ 0.0 & -5.0 & 3.0 & -2.0 \\ 0.0 & 0.0 & -2.1999999999999993 & -2.2 \end{bmatrix} \tag{3}$$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \in \mathbb{R}^{3\times 3}, \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \in \mathbb{R}^3, \boldsymbol{c} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \in \mathbb{R}^3.$$

$$[\ \boldsymbol{A}\ \ \boldsymbol{b}\ ] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 3 \end{array}\right] \sim [\ \boldsymbol{A}_1\ \ \boldsymbol{b}_1\ ] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 & = 3 \\ x_2 + x_3 & = 1 \\ 0 & = 0 \end{cases},$$

$$[\ \boldsymbol{A}\ \ \boldsymbol{c}\ ] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{array}\right] \sim [\ \boldsymbol{A}_1\ \ \boldsymbol{c}_1\ ] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \Leftrightarrow \begin{cases} x_1 + 2x_2 + 3x_3 & = 3 \\ x_2 + x_3 & = 1 \\ 0 & = 1 \end{cases}.$$

- Use similarity transformations to *reduced row echelon form*:
  - All zero rows are below non-zero rows
  - First non-zero entry on a row is called the *leading entry*
  - In each non-zero row, the leading entry is to the left of lower leading entries
  - Each leading entry equals 1 and is the only non-zero entry in its column
- *Row echelon form*:
  - Allow additional non-zero elements in a column, above the leading entry
- After carrying out rref on bordered matrix $[\ \boldsymbol{A}\ \mid\ \boldsymbol{b}\ ]$, if:
  - there is a row with $[\ 0\ \ 0\ \ \dots\ \ 0\ \mid\ 1\ ] \Rightarrow$ No solutions
  - the result is of form $[\ \boldsymbol{I}\ \mid\ \boldsymbol{c}\ ] \Rightarrow$ Unique solution
  - there is no row of form $[\ 0\ \ 0\ \ \dots\ \ 0\ \mid\ 1\ ]$, and there is a row of all zeros $[\ 0\ \ 0\ \ \dots\ \ 0\ \mid\ 0\ ] \Rightarrow$ Infinitely many solutions

  Examples

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \text{Infinitely many solutions}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -4 & -4 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \text{Unique solution}$$

- Recall the basic operation in row echelon reduction: constructing a linear combination of rows to form zeros beneath the main diagonal, e.g.

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} \sim \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} a_{12} & \dots & a_{2m} - \frac{a_{21}}{a_{11}} a_{1m} \\ 0 & a_{32} - \frac{a_{31}}{a_{11}} a_{12} & \dots & a_{3m} - \frac{a_{31}}{a_{11}} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - \frac{a_{m1}}{a_{11}} a_{12} & \dots & a_{mm} - \frac{a_{m1}}{a_{11}} a_{1m} \end{pmatrix}$$

- This can be stated as a matrix multiplication operation, with $l_{i1} = a_{i1} / a_{11}$

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -l_{21} & 1 & 0 & \dots & 0 \\ -l_{31} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -l_{m1} & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ a_{31} & a_{32} & \dots & a_{3m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ 0 & a_{22} - l_{21} a_{12} & \dots & a_{2m} - l_{21} a_{1m} \\ 0 & a_{32} - l_{31} a_{12} & \dots & a_{3m} - l_{31} a_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2} - l_{m1} a_{12} & \dots & a_{mm} - l_{m1} a_{1m} \end{pmatrix}$$

- Denote a permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & m \\ i_1 & i_2 & \cdots & i_m \end{pmatrix}$$

  with $i_1, \ldots, i_m \in \{1, \ldots, m\}$, $i_j \neq i_k$ for $j \neq k$

- The sign of a permutation, $\nu(\sigma)$ is the number of pair swaps needed to obtain the permutation starting from the identity permutation

$$\begin{pmatrix} 1 & 2 & \ldots & m \\ 1 & 2 & \cdots & m \end{pmatrix}$$

- A permutation can be specified by a permutation matrix $\boldsymbol{P}$ obtained from $\boldsymbol{I}$ by swapping rows and columns $k \leftrightarrow i_k$

Definition. *The matrix*

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 1 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ 0 & \dots & -l_{k+2,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

with $l_{i,k} = a_{i,k}^{(k)} / a_{k,k}^{(k)}$, *and* $\boldsymbol{A}^{(k)} = \left(a_{i,j}^{(k)}\right)$ *the matrix obtained after step* k *of row echelon reduction (or, equivalently, Gaussian elimination) is called a Gaussian multiplier matrix.*

Permutation and Gaussian multiplier matrices are *elementary matrices*.

- The Gaussian multiplier matrix ...

$$\boldsymbol{L}_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & -l_{k+1,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & -l_{m,k} & \dots & 1 \end{pmatrix}$$

- ... has inverse (matrix that "undoes" the linear transformation)

$$\boldsymbol{L}_k^{-1} = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & \dots & 1 & \dots & 0 \\ 0 & \dots & l_{k+1,k} & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots \\ 0 & \dots & l_{m,k} & \dots & 1 \end{pmatrix}$$

- Consider elementary matrices

$$\boldsymbol{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \boldsymbol{E}_1\boldsymbol{E}_2 = \boldsymbol{E}_2\boldsymbol{E}_1 = \boldsymbol{I},$$

  stating that $\boldsymbol{E}_1$ undoes the effect of $\boldsymbol{E}_2$.

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ is invertible if there exists $\boldsymbol{X} \in \mathbb{R}^{m \times m}$ such that

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{A} = \boldsymbol{I}$$

- Notation $\boldsymbol{X} = \boldsymbol{A}^{-1}$, is the *inverse* of $\boldsymbol{A}$.

- What about general square matrices $\boldsymbol{A} \in \mathbb{R}^{m \times m}$? How to find inverse
- $\boldsymbol{X}$ is inverse if $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}$ or

$$\boldsymbol{A}[\ \boldsymbol{x}_1 \ \ \boldsymbol{x}_2 \ \ \dots \ \ \boldsymbol{x}_m\ ] = [\ \boldsymbol{A}\boldsymbol{x}_1 \ \ \boldsymbol{A}\boldsymbol{x}_2 \ \ \dots \ \ \boldsymbol{A}\boldsymbol{x}_m\ ] = [\ \boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \dots \ \ \boldsymbol{e}_m\ ]$$

- Find the inverse is equivalent to solving systems $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{e}_1, \dots, \boldsymbol{A}\boldsymbol{x}_m = \boldsymbol{e}_m$
- Gauss Jordan algoritm generalizes Gaussian elimination that solves a single linear system to solving m systems simultaneously by forming the bordered matrix $[\ \boldsymbol{A} \ | \ \boldsymbol{I}\ ]$

$$[\ \boldsymbol{A} \ | \ \boldsymbol{I}\ ] \sim [\ \boldsymbol{I} \ | \ \boldsymbol{X}\ ]$$

- When does a matrix inverse exist? $\boldsymbol{A} \in \mathbb{R}^{m \times m}$

  a $\boldsymbol{A}$ invertible
  b $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a unique solution for all $\boldsymbol{b} \in \mathbb{R}^m$
  c $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ has a unique solution
  d The reduced row echelon form of $\boldsymbol{A}$ is $\boldsymbol{I}$
  e $\boldsymbol{A}$ can be written as product of elementary matrices

$$a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$$

$a \Rightarrow b$ $\boldsymbol{A}$ invertible $\Rightarrow$ $\boldsymbol{A}^{-1}$ exists, and $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$ is a solution $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{b}) = (\boldsymbol{A}\boldsymbol{A}^{-1})\boldsymbol{b} = \boldsymbol{b}$. If there were two solutions $\boldsymbol{x}, \boldsymbol{y}$, then

$$\boldsymbol{x} - \boldsymbol{y} = (\boldsymbol{A}^{-1}\boldsymbol{A})(\boldsymbol{x} - \boldsymbol{y}) = \boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{A}\boldsymbol{y}) = \boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{b}) = \boldsymbol{A}^{-1}\boldsymbol{0} = \boldsymbol{0}.$$

$b \Rightarrow c$ Choose $\boldsymbol{b} = \boldsymbol{0}$

$c \Rightarrow d$ $[\ \boldsymbol{A} \mid \boldsymbol{0}\ ] \sim [\ \boldsymbol{U} \mid \boldsymbol{0}\ ]$. If $\boldsymbol{U} \neq \boldsymbol{I}$ there is a row of zeros, and solution is not unique. If solution is unique then $\boldsymbol{U} = \boldsymbol{I}$

$d \Rightarrow e$ $[\ \boldsymbol{A} \mid \boldsymbol{0}\ ] \sim [\ \boldsymbol{I} \mid \boldsymbol{0}\ ]$ implies $\boldsymbol{E}_k \ldots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I} \Rightarrow \boldsymbol{A} = \boldsymbol{E}_1^{-1} \ldots \boldsymbol{E}_k^{-1}$

$e \Rightarrow a$ $\boldsymbol{A} = \boldsymbol{E}_1^{-1} \ldots \boldsymbol{E}_k^{-1} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{E}_k \ldots \boldsymbol{E}_1$.

- The inverse of a product $(\boldsymbol{A}\boldsymbol{B})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

$$(\boldsymbol{A}\boldsymbol{B})\boldsymbol{B}^{-1}\boldsymbol{A}^{-1} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{B}^{-1})\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{I}\boldsymbol{A}^{-1} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$$

$$\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}(\boldsymbol{A}\boldsymbol{B}) = \boldsymbol{B}^{-1}(\boldsymbol{A}^{-1}\boldsymbol{A})\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{I}\boldsymbol{B} = \boldsymbol{B}^{-1}\boldsymbol{B} = \boldsymbol{I}$$

- If $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ invertible so are: $c\boldsymbol{A}$, $\boldsymbol{A}^T$, $\boldsymbol{A}^k$

$$(\boldsymbol{A}^T)^{-1} = (\boldsymbol{A}^{-1})^T$$

Verify

$$\boldsymbol{A}^T(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^{-1}\boldsymbol{A})^T = \boldsymbol{I}$$

$$(\boldsymbol{A}^{-1})^T\boldsymbol{A}^T = (\boldsymbol{A}\boldsymbol{A}^{-1})^T = \boldsymbol{I}$$