Overview

- Determinants
  - geometric interpretation
  - computation rules
- Characteristic polynomial
  - repeated roots, algebraic multiplicity
- Eigenspaces
  - null space dimension, geometric multiplicity
- Eigendecomposition
  - possible if algebraic multiplicity equals geometric multiplicity for each eigenvalues
  - simple, meaning, an orthogonal or unitary decomposition for normal matrices
- Computing the SVD reduces to computing two eigenproblems

Definition. *The determinant of a square matrix* $\boldsymbol{A} = [\,\boldsymbol{a}_1 \;\; \dots \;\; \boldsymbol{a}_m\,] \in \mathbb{R}^{m\times m}$

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{vmatrix} \in \mathbb{R}$$

is a real number giving the (oriented) volume of the parallelepiped spanned by matrix column vectors.

- $m = 2$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

- $m = 3$

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \det(\boldsymbol{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

- Computation of a determinant with $m=2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

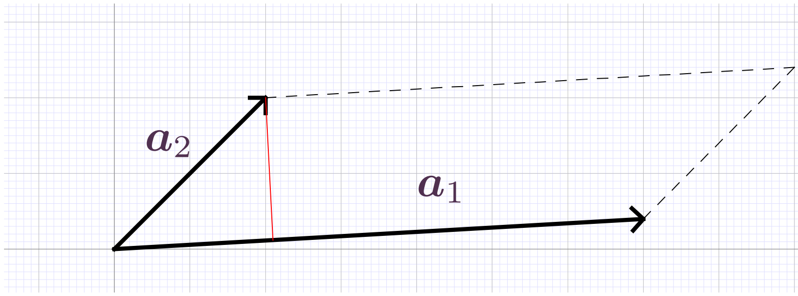
- Computation of a determinant with $m=3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$

$$-a_{13}a_{22}a_{31} - a_{23}a_{32}a_{11} - a_{33}a_{12}a_{21}$$

- Where do these determinant computation rules come from? Two viewpoints
  - *Geometric viewpoint*: determinants express parallelepiped volumes
  - *Algebraic viewpoint*: determinants are computed from all possible products that can be formed from choosing a factor from each row and each column

- $m = 2$



$$\boldsymbol{A} = [\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \,] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

- In two dimensions a "parallelepiped" becomes a parallelogram with area given as

$$(\text{Area}) = (\text{Length of Base}) \times (\text{Length of Height})$$

- Take $\boldsymbol{a}_1$ as the base, with length $b = \|\boldsymbol{a}_1\|$. Vector $\boldsymbol{a}_1$ is at angle φ_1 to x_1-axis, $\boldsymbol{a}_2$ is at angle φ_2 to x_2-axis, and the angle between $\boldsymbol{a}_1$, $\boldsymbol{a}_2$ is $\theta = \varphi_2 - \varphi_1$. The height has length

$$h = \|\boldsymbol{a}_2\| \sin\theta = \|\boldsymbol{a}_2\| \sin(\varphi_2 - \varphi_1) = \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2)$$

- Use $\cos\varphi_1 = a_{11} / \|\boldsymbol{a}_1\|$, $\sin\varphi_1 = a_{12} / \|\boldsymbol{a}_1\|$, $\cos\varphi_2 = a_{21} / \|\boldsymbol{a}_2\|$, $\sin\varphi_2 = a_{22} / \|\boldsymbol{a}_2\|$

$$(\text{Area}) = \|\boldsymbol{a}_1\| \, \|\boldsymbol{a}_2\| (\sin\varphi_2 \cos\varphi_1 - \sin\varphi_1 \cos\varphi_2) = a_{11}a_{22} - a_{12}a_{21}$$

- The geometric interpretation of a determinant as an oriented volume is useful in establishing rules for calculation with determinants:
  - Determinant of matrix with repeated columns is zero (since two edges of the parallelepiped are identical). Example for $m=3$

    $$\Delta = \begin{vmatrix} a & a & u \\ b & b & v \\ c & c & w \end{vmatrix} = abw + bcu + cav - ubc - vca - wab = 0$$

    This is more easily seen using the column notation

    $$\Delta = \det\left(\begin{array}{cccc} \boldsymbol{a}_1 & \boldsymbol{a}_1 & \boldsymbol{a}_3 & \ldots \end{array} \right) = 0$$

  - Determinant of matrix with linearly dependent columns is zero (since one edge lies in the 'hyperplane' formed by all the others)

- Separating sums in a column (similar for rows)

$$\det(\ \boldsymbol{a}_1+\boldsymbol{b}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )=\det(\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )+\det(\ \boldsymbol{b}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )$$

  with $\boldsymbol{a}_i, \boldsymbol{b}_1 \in \mathbb{R}^m$

- Scalar product in a column (similar for rows)

$$\det(\ \alpha\boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )=\alpha\det(\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )$$

  with $\alpha \in \mathbb{R}$

- Linear combinations of columns (similar for rows)

$$\det(\ \boldsymbol{a}_1\ \ \boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )=\det(\ \boldsymbol{a}_1\ \ \alpha\boldsymbol{a}_1+\boldsymbol{a}_2\ \ \ldots\ \ \boldsymbol{a}_m\ )$$

  with $\alpha \in \mathbb{R}$.

- $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$
- $\det(\boldsymbol{A}\boldsymbol{B}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, if $\operatorname{rank}(\boldsymbol{A}) < m$ then $\det(\boldsymbol{A}) = 0$

- For square matrix $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ find *non-zero* vectors whose *directions* are not changed by multiplication by $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$, λ is scalar, the *eigenvalue problem*.
- Consider the eigenproblem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ for $\boldsymbol{A} \in \mathbb{R}^{m \times m}$. Rewrite as

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x} \Rightarrow (\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}.$$

  Since $\boldsymbol{x} \neq \boldsymbol{0}$, a solution to eigenproblem exists only if $\boldsymbol{A} - \lambda \boldsymbol{I}$ is singular.
- $\boldsymbol{A} - \lambda \boldsymbol{I}$ singular implies $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$.
- Investigate form of $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \dots & a_{1m} \\ a_{21} & a_{22} - \lambda & a_{23} & \dots & a_{2m} \\ a_{31} & a_{32} & a_{33} - \lambda & \dots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} - \lambda \end{vmatrix}$$

- $p_m(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$, an m^{th}-degree polynomial in λ, *characteristic polynomial* of $\boldsymbol{A}$, with m roots, λ_1, $\lambda_2, \dots, \lambda_m$, the eigenvalues of $\boldsymbol{A}$

- $\boldsymbol{A} \in \mathbb{R}^{m \times m}$, eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$ ($\boldsymbol{x} \neq \boldsymbol{0}$) in matrix form:

$$\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$$

$$\boldsymbol{X} = [\ \boldsymbol{x}_1 \ \ \dots \ \ \boldsymbol{x}_m\ ], \boldsymbol{\Lambda} = \operatorname{diag}(\lambda_1, \dots, \lambda_m) = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_m \end{bmatrix}.$$

- $\boldsymbol{X}$ is the *eigenvector matrix*, $\boldsymbol{\Lambda}$ is the (diagonal) *eigenvalue matrix*
- If column vectors of $\boldsymbol{X}$ are linearly independent, then $\boldsymbol{X}$ is invertible

$$\boldsymbol{A} = \boldsymbol{X}\boldsymbol{\Lambda}\boldsymbol{X}^{-1},$$

  the *eigendecomposition* of $\boldsymbol{A}$ (compare to $\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$, $\boldsymbol{A} = \boldsymbol{Q}\boldsymbol{R}$)
- Rule "determinant of product = product of determinants" implies

$$\det(\boldsymbol{A}\boldsymbol{X}) = \det(\boldsymbol{X}\boldsymbol{\Lambda}) \Rightarrow \det(\boldsymbol{A}) = \det(\boldsymbol{\Lambda}) \text{ (for } \det(X) \neq 0).$$

- Eigendecomposition of $\boldsymbol{I} \in \mathbb{R}^{m \times m}$. Compare $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$

$$\boldsymbol{I}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{I}, \boldsymbol{A} = \boldsymbol{I}, \boldsymbol{X} = \boldsymbol{I}, \boldsymbol{\Lambda} = \boldsymbol{I}$$

  to find eigenvalues $\lambda_1 = 1, \ldots, \lambda_m = 1$, eigenvectors $\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{x}_m = \boldsymbol{e}_m$.

- Eigendecomposition of $\boldsymbol{A} = \operatorname{diag}(s_1, s_2, \ldots, s_m)$. Compare $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}$

$$\boldsymbol{A}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{A}$$

  to find eigenvalues $\lambda_1 = s_1, \ldots, \lambda_m = s_m$, eigenvectors $\boldsymbol{x}_1 = \boldsymbol{e}_1, \ldots, \boldsymbol{x}_m = \boldsymbol{e}_m$.

- Reflection across x_1-axis in $\mathbb{R}^2$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

  is a diagonal matrix, $\lambda_1 = 1$, $\lambda_2 = -1$, $\boldsymbol{x}_1 = \boldsymbol{e}_1$, $\boldsymbol{x}_2 = \boldsymbol{e}_2$

- Rotate by θ around x_3 axis in $\mathbb{R}^3$

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3}, m = 3$$

- One direction not change by rotation is $\boldsymbol{x}_3 = \boldsymbol{e}_3$ with $\lambda_3 = 1$
- Where are the other two directions?
  - Compute characteristic polynomial $p_3(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$

$$p_3(\lambda) = \begin{vmatrix} \lambda - \cos\theta & \sin\theta & 0 \\ -\sin\theta & \lambda - \cos\theta & 0 \\ 0 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)(\lambda^2 - 2\lambda\cos\theta + 1)$$

  - One root of $p_3(\lambda)$ is $\lambda_3 = 1$, as expected.
  - Solve $\lambda^2 - 2\lambda\cos\theta + 1 = 0$ to find remaining eigenvalues to be *complex*

$$\lambda_{1,2} = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta} \in \mathbb{C}, i^2 = -1.$$

- $z \in \mathbb{C}$ can be represented in
  - Cartesian form $z = x + iy$
  - Polar form $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$
- Complex conjugate of $z \in \mathbb{C}$ negates imaginary part $\bar{z} = x - iy = re^{-i\theta}$
- Absolute value of $z \in \mathbb{C}$ is $|z| = (x^2 + y^2)^{1/2} = r$
- Argument of z is angle θ from polar form $z = re^{i\theta}$
- Absolute value can be expressed as $|z| = (\bar{z}\,z)^{1/2}$
- Recall for $\boldsymbol{x} \in \mathbb{R}^m$

$$\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^T\boldsymbol{x} = x_1^2 + \cdots + x_m^2,$$

  stating that squared 2-norm of real vector $\boldsymbol{x}$ is sum of squares of components.
- Extend above to vector of complex numbers $\boldsymbol{u} \in \mathbb{C}^m$ by

$$\|\boldsymbol{u}\|_2^2 = |u_1|^2 + \cdots + |u_m|^2 = (\bar{\boldsymbol{u}})^T\boldsymbol{u}.$$

- Taking the complex conjugate and transposing arises frequently, notation

$$\boldsymbol{u}^* = (\bar{\boldsymbol{u}})^T, \text{adjoint of } \boldsymbol{u}$$

- Consider $\lambda_2 = e^{i\theta} = \exp(i\theta) = \cos\theta + i\sin\theta$. Eigenvector $\boldsymbol{x}_2$ satisfies

$$(\boldsymbol{A} - \lambda_2 \boldsymbol{I})\boldsymbol{x}_2 = \boldsymbol{0},$$

which implies $\boldsymbol{x}_2 \in N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$

- Compute basis vector for $N(\boldsymbol{A} - \lambda_2 \boldsymbol{I})$

$$\boldsymbol{A} - \lambda_2 \boldsymbol{I} = \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix} \sim \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix}.$$

- Find eigenvector

$$\boldsymbol{x}_2 = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

- Repeat for $\lambda_2 = e^{-i\theta}$, find $\boldsymbol{x}_3 = \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix}$

- Compute $\boldsymbol{A}\boldsymbol{x}_2 - \lambda_2\boldsymbol{x}_2 = (\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{x}_2$

$$(\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{x}_2 = \begin{bmatrix} -i\sin\theta & -\sin\theta & 0 \\ \sin\theta & -i\sin\theta & 0 \\ 0 & 0 & -e^{i\theta} \end{bmatrix}\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \boldsymbol{0}. \checkmark$$

- In general a polynomial of degree m $p_m(\lambda)$ with real coefficients has m complext roots $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$

- Consider

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Eigenvalues $\lambda_1 = \lambda_2 = 1$, *a repeated root*, since

$$\det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 1 & 1 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2.$$

- However

$$\boldsymbol{A} - \lambda_1 \boldsymbol{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\operatorname{rank}(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 1 = \dim C((\boldsymbol{A} - \lambda_1 \boldsymbol{I})^T)$, FTLA $\Rightarrow \dim N(\boldsymbol{A} - \lambda_1 \boldsymbol{I}) = 1$, only one non-zero eigenvector

Definition 1. *The algebraic multiplicity of an eigenvalue λ is the number of times it appears as a repeated root of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$*

Example. $p(\lambda) = \lambda(\lambda - 1)(\lambda - 2)^2$ has two single roots $\lambda_1 = 0$, $\lambda_2 = 1$ and a repeated root $\lambda_{3,4} = 2$. The eigenvalue $\lambda = 2$ has an algebraic multiplicity of 2

Definition 2. *The geometric multiplicity of an eigenvalue λ is the dimension of the null space of $A - \lambda I$*

Definition 3. *An eigenvalue for which the geometric multiplicity is less than the algebraic multiplicity is said to be defective*

Theorem. *A matrix is diagonalizable if the geometric multiplicity of each eigenvalue is equal to the algebraic multiplicity of that eigenvalue.*

- Find eigenvectors as non-trivial solutions of system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x}=0$, e.g., $\lambda_1=1$

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} 4 & -4 & 2 \\ 5 & -5 & 1 \\ -2 & 2 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 5 & -5 & 1 \end{pmatrix} \sim \begin{pmatrix} -2 & 2 & -4 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

  Note convenient choice of row operations to reduce amount of arithmetic, and use of knowledge that $\boldsymbol{A}-\lambda_1\boldsymbol{I}$ is singular to deduce that last row must be null

- In traditional form the above row-echelon reduced system corresponds to

$$\begin{cases} -2x_1+2x_2-4x_3 & = & 0 \\ 0x_1+0x_2-6x_3 & = & 0 \\ 0x_1+0x_2+0x_3 & = & 0 \end{cases} \Rightarrow \boldsymbol{x}=\alpha\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \|\boldsymbol{x}\|=1 \Rightarrow \alpha=1/\sqrt{2}$$

- Suppose $\boldsymbol{A} \in \mathbb{R}^{m \times m}$ diagonalizable, $\boldsymbol{A} = \boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1}$
- Repeated application of $\boldsymbol{A}$

$$\boldsymbol{A}^2 = (\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1})(\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1}) = \boldsymbol{X}\,\boldsymbol{\Lambda}^2\,\boldsymbol{X}^{-1}$$

$$\boldsymbol{A}^k = (\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1}) \cdot \dots \cdot (\boldsymbol{X}\,\boldsymbol{\Lambda}\,\boldsymbol{X}^{-1}) = \boldsymbol{X}\,\boldsymbol{\Lambda}^k\,\boldsymbol{X}^{-1}$$

- Above allows definition of $e^{\boldsymbol{A}}, \sin(\boldsymbol{A}), \cos(\boldsymbol{A})$, for example

$$e^x = \frac{1}{0!}x^0 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \dots + \frac{1}{k!}x^k + \dots \Rightarrow$$

$$e^{\boldsymbol{A}} = \boldsymbol{X}\left(\frac{1}{0!}\boldsymbol{\Lambda}^0 + \frac{1}{1!}\boldsymbol{\Lambda} + \frac{1}{2!}\boldsymbol{\Lambda}^2 + \dots + \frac{1}{k!}\boldsymbol{\Lambda}^k + \dots\right)\boldsymbol{X}^{-1}$$

- The differential system $\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y}$ has solution $\boldsymbol{y}(t) = e^{\boldsymbol{A}t}\,y(0)$.