Homework 5

Due date: Feb 13, 2020, 11:55PM.

Bibliography: Lesson07.pdf Lesson08.pdf. The first exercise in each problem set is solved for you to use as a model.

1. Consider $\mathcal{V}=\mathbb{R}^3$, $\mathcal{S}=\mathbb{R}$. Establish whether for given operations $\oplus,\odot$ $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is a vector space or not.

Ex 1. $\boldsymbol{x}=(x_1,x_2,x_3)$, $\boldsymbol{y}=(y_1,y_2,y_3)$, $\boldsymbol{x},\boldsymbol{y}\in\mathcal{V}=\mathbb{R}^3$, $\boldsymbol{x}\oplus\boldsymbol{y}=(2x_1+2y_1,2x_2+2y_2,2x_3+2y_3)$, and with $\alpha\in\mathcal{S}=\mathbb{R}$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$

Solution. Check associativity, $(\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}=\boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})$. Compute

$$\boldsymbol{u}=\boldsymbol{x}\oplus\boldsymbol{y}=(2x_1+2y_1,2x_2+2y_2,2x_3+2y_3)\quad \boldsymbol{v}=\boldsymbol{y}\oplus\boldsymbol{z}=(2y_1+2z_1,2y_2+2z_2,2y_3+2z_3)$$

$$\boldsymbol{u}+\boldsymbol{z}=(4x_1+4y_1+2z_1,4x_2+4y_2+2z_2,4x_3+4y_3+2z_2)$$

$$\boldsymbol{x}+\boldsymbol{v}=(2x_1+4y_1+4z_1,2x_2+4y_2+4z_2,2x_2+4y_2+4z_2)$$

Since $\boldsymbol{u}+\boldsymbol{z}\neq\boldsymbol{x}+\boldsymbol{v}$, associativity is not satisified and $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is not a vector space. Note: it is sufficient to find one unsatisfied property to prove that $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is not a vector space. However to prove that $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is indeed a vector space, all properties must be verified/

Ex 2. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1-y_1,x_2-y_2,x_3-y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Ex 3. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Ex 4. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1,x_2-y_2,x_3+y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Ex 5. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1,x_2+y_2,x_3+y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha+x_1,\alpha x_2,\alpha x_3)$.

2. Consider $\mathcal{V}\subset\mathbb{R}^{2\times2}$, a subset of all 2 by 2 real-component matrices with operations

$$\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times2},\boldsymbol{A}=\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},\boldsymbol{B}=\begin{pmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix},\boldsymbol{A}\oplus\boldsymbol{B}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12}\\ a_{21}+b_{21} & a_{22}+b_{22}\end{pmatrix}$$

$$\alpha\odot\boldsymbol{A}=\begin{pmatrix} \alpha a_{11} & \alpha a_{12}\\ \alpha a_{21} & \alpha a_{22}\end{pmatrix}.$$

Determine whether the following are vector spaces

Ex 1. $\mathcal{V}$ is the set of skew-symmetric matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ \boldsymbol{A}^T=-\boldsymbol{A}$.

Solution. From $\boldsymbol{A}=-\boldsymbol{A}^T$ deduce that

$$\begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}=-\begin{pmatrix} a_{11} & a_{21}\\ a_{12} & a_{22}\end{pmatrix}\Rightarrow\boldsymbol{A}=\begin{pmatrix} 0 & a\\ -a & 0\end{pmatrix}.$$

Verify vector space properties for $\forall\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\in\mathcal{V}$, $\forall\alpha,\beta\in\mathbb{R}$:

Closure.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} 0 & a\\ -a & 0\end{pmatrix}+\begin{pmatrix} 0 & b\\ -b & 0\end{pmatrix}=\begin{pmatrix} 0 & a+b\\ -(a+b) & 0\end{pmatrix}\in\mathcal{V}\checkmark$$

Associativity.

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\begin{pmatrix} 0 & a+b\\ -(a+b) & 0\end{pmatrix}+\begin{pmatrix} 0 & c\\ -c & 0\end{pmatrix}=\begin{pmatrix} 0 & a+b+c\\ -(a+b+c) & 0\end{pmatrix}$$

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix} 0 & a\\ -a & 0\end{pmatrix}+\begin{pmatrix} 0 & b+c\\ -(b+c) & 0\end{pmatrix}=\begin{pmatrix} 0 & a+b+c\\ -(a+b+c) & 0\end{pmatrix}.\checkmark$$

Identity.

$$\boldsymbol{A}+\boldsymbol{0}=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}\checkmark$$

Inverse.

$$\boldsymbol{A}+(-\boldsymbol{A})=\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}+\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\checkmark$$

Commutativity.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} 0 & a+b \\ -(a+b) & 0 \end{pmatrix}=\begin{pmatrix} 0 & b+a \\ -(b+a) & 0 \end{pmatrix}=\boldsymbol{B}+\boldsymbol{A}.\checkmark$$

Distributivity.

$$\alpha(\boldsymbol{A}+\boldsymbol{B})=\begin{pmatrix} 0 & \alpha(a+b) \\ -\alpha(a+b) & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha a+\alpha b) \\ -\alpha a-\alpha b) & 0 \end{pmatrix}=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}\checkmark$$

$$(\alpha+\beta)\boldsymbol{A}=\begin{pmatrix} 0 & (\alpha+\beta)a \\ -(\alpha+\beta)a & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha a+\beta a \\ -\alpha a-\beta a & 0 \end{pmatrix}=\alpha\boldsymbol{A}+\beta\boldsymbol{A}\checkmark$$

$$\alpha(\beta\boldsymbol{A})=\alpha\begin{pmatrix} 0 & \beta a \\ -\beta a & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha\beta a \\ -\alpha\beta a & 0 \end{pmatrix}=(\alpha\beta)\boldsymbol{A}.\checkmark$$

All properties are verified, hence skew-symmetric matrices form a vector space.

Ex 2. $\mathcal{V}$ is the set of upper-triangular matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ a_{21}=0$.

Ex 3. $\mathcal{V}$ is the set of symmetric matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ \boldsymbol{A}^T=\boldsymbol{A}$.

3. Determine whether the set $\mathcal{S}$ is linearly dependent or independent within the vector space $\mathcal{V}$

Ex 1.

$$\mathcal{S}=\{\boldsymbol{u}_1,\boldsymbol{u}_2\}=\left\{\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\},\mathcal{V}=\mathbb{R}^3$$

Solution. The first equation of the system $a_1\boldsymbol{u}_1+a_2\boldsymbol{u}_2=\boldsymbol{0}$ is $2a_1+0a_2=0\Rightarrow a_1=0$. The second equation then states $-a_2=0$, hence $a_1=a_2=0$, and $\boldsymbol{u}_1,\boldsymbol{u}_2$ are linearly independent.

Ex 2.

$$\mathcal{S}=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}=\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix},\begin{pmatrix} 1 \\ 0 \end{pmatrix},\begin{pmatrix} 8 \\ -3 \end{pmatrix}\right\},\mathcal{V}=\mathbb{R}^2$$

Ex 3.

$$\mathcal{S}=\{\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3\}=\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\},\mathcal{V}=\mathbb{R}^3$$

4. Determine whether the set $\mathcal{S}$ is linearly dependent or independent within the vector space $\mathcal{V}$. Here $\mathcal{P}_n$ is the set of polynomials of degree at most n.

Ex 1.

$$\mathcal{S}=\{\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_3\}=\{1,2x^2+x+2,-x^2+x\},\mathcal{V}=\mathcal{P}_2$$

Solution. Denote $\boldsymbol{q}=a_1\boldsymbol{p}_1+a_2\boldsymbol{p}_2+a_3\boldsymbol{p}_3$, and consider the equality $\boldsymbol{q}=\boldsymbol{0}$. Note that $\boldsymbol{0}$ is the zero polynomial, i.e. $\boldsymbol{q}(x)=\boldsymbol{0}(x)\Rightarrow$

$$a_1+a_2(2x^2+x+2)+a_3(-x^2+x)=0\,\text{for all}\,x$$

For $x=0$ obtain $a_1+a_2=0$. Subsequently for $x=1$ obtain $a_1+5a_2=0$. Subtract to obtain $4a_2=0\Rightarrow a_2=0$, and then $a_1=0$. Then for $x=-1$ obtain $-2a_3=0\Rightarrow a_3=0$. The only choice of a_1,a_2,a_3 to have $a_1\boldsymbol{p}_1+a_2\boldsymbol{p}_2+a_3\boldsymbol{p}_3=\boldsymbol{0}$ is $a_1=a_2=a_3=0$, hence $\mathcal{S}$ is a linearly independent set of vectors.

Ex 2.

$$\mathcal{S} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\} = \{2, x, x^3 + 2x^2 - 1\}, \mathcal{V} = \mathcal{P}_3$$

Ex 3.

$$\mathcal{S} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\} = \{x, x^2, x^2 + 2x, x^3 - x + 1\}, \mathcal{V} = \mathcal{P}_3$$