Homework 6

Due date: Feb 20, 2020, 11:55PM.

Bibliography: Lesson09.pdf Lesson10.pdf. The first exercise in each problem set is solved for you to use as a model.

1. Consider $\mathcal{V}=\mathbb{R}^n$, $\mathcal{S}=\mathbb{R}$. Determine whether the column vectors of $\boldsymbol{A}$ form a basis for $\mathbb{R}^n$.

Ex 1. $n=3$,

$$\boldsymbol{A}=\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Solution. Reduce $\boldsymbol{A}$ to row-echelon form

$$\boldsymbol{A}\sim\begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 2 & 2 \end{pmatrix}\sim\begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}\sim\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Since the row-echelon form does not contain a row of zeros, the columns of $\boldsymbol{A}$ form a basis for $\mathbb{R}^3$. Check in Maxima

```
(%i1) A: matrix([-1, 1, 1],[2, 0, 1],[1, 1, 1]);
```

$$(\%o1)\quad \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

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(%i2) echelon(A);
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$$(\%o2)\quad \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

```
(%i3)
```

Note: multiple row-echelon forms are possible (differing by, say, permutation of rows), but the same conclusion on the idependence is reached, i.e., no zero rows implies linear independence.

Ex 2. $n=3$,

$$\boldsymbol{A}=\begin{pmatrix} 2 & 5 & 3 \\ -2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Ex 3. $n=4$,

$$\boldsymbol{A}=\begin{pmatrix} 1 & 2 & 2 & -1 \\ 1 & 1 & 4 & 2 \\ -1 & 3 & 2 & 0 \\ 1 & 1 & 5 & 3 \end{pmatrix}.$$

Ex 4. $n=4$,

$$\boldsymbol{A}=\begin{pmatrix} -1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 0 & -1 & 1 & 1 \\ 1 & 2 & -1 & 2 \end{pmatrix}.$$

2. Determine whether the following column vectors of $\boldsymbol{B}$ form a basis for $(\mathcal{V},+,\mathbb{R},\cdot)$

Ex 1. $\boldsymbol{B}=\begin{pmatrix} 1 & 2x^2+x+2 & -x^2+x \end{pmatrix}$, $\mathcal{V}=\mathcal{P}_2$

Solution. Consider an arbitrary $\boldsymbol{p} \in \mathcal{V} = \mathcal{P}_2$

$$\boldsymbol{p}(x) = a_0 + a_1 x + a_2 x^2,$$

and check if $\boldsymbol{p}$ can be expressed as a linear combination of the basis vectors, $\boldsymbol{B} =$, i.e., there exists $\boldsymbol{c} \in \mathbb{R}^3$ such that $\boldsymbol{B}\boldsymbol{c} = \boldsymbol{p}$

$$\boldsymbol{B}\boldsymbol{c} = \begin{pmatrix} \boldsymbol{b}_1(x) & \boldsymbol{b}_2(x) & \boldsymbol{b}_3(x) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = c_1 \boldsymbol{b}_1(x) + c_2 \boldsymbol{b}_2(x) + c_3 \boldsymbol{b}_3(x) = a_0 + a_1 x + a_2 x^2 = \boldsymbol{p}(x).$$

Identify powers of x

$$c_1 \boldsymbol{b}_1(x) + c_2 \boldsymbol{b}_2(x) + c_3 \boldsymbol{b}_3(x) = c_1 + c_2(2x^2 + x + 2) + c_3(-x^2 + x) = (c_1 + 2c_2) \cdot 1 + (c_1 + c_2) \cdot x + (2c_2 - c_3) \cdot x^2$$

to obtain system $\boldsymbol{M}\boldsymbol{c} = \boldsymbol{a}$

$$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix},$$

Reduce matrix $\boldsymbol{M}$ to row-echelon form.

$$\boldsymbol{M} = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the row-echelon form of $\boldsymbol{M}$ does not have any zero rows, a unique solution of $\boldsymbol{M}\boldsymbol{c} = \boldsymbol{a}$ is found, and $\boldsymbol{B}$ is a basis for $\mathcal{P}_3$. Check in Maxima

```
(%i3) M: matrix([1, 2, 0],[1, 1, 0],[0, 2, -1]);
```

(%o3) $$\begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & -1 \end{pmatrix}$$

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(%i4) echelon(M);
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(%o4) $$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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(%i5)
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Ex 2. $\mathcal{V} = \mathbb{R}^{2\times 2}$ the space of real-valued two-by-two matrices,

$$\boldsymbol{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \right)$$

Ex 3. $\mathcal{V}$ is the set of symmetric matrices, $\boldsymbol{A} \in \mathcal{V} \Rightarrow \boldsymbol{A}^T = \boldsymbol{A}$.

$$\boldsymbol{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

3. Find a basis for the subspace $\mathcal{S}$ of vector space $\mathcal{V}$. Specify the dimension of $\mathcal{S}$.

Ex 1.

$$\mathcal{S} = \left\{ \begin{pmatrix} s + 2t \\ -s + t \\ t \end{pmatrix} \middle| \, s, t \in \mathbb{R} \right\}, \mathcal{V} = \mathbb{R}^3$$

Solution. Recognize that the specification of $\mathcal{S}$ gives a linear combination with scalar coefficients s,t, and rewrite

$$\begin{pmatrix} s+2t \\ -s+t \\ t \end{pmatrix} = s\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Construct

$$\boldsymbol{B} = (\ \boldsymbol{b}_1\ \ \boldsymbol{b}_2\ ) = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Check if columns of $\boldsymbol{B}$ are linearly independent,

$$s\boldsymbol{b}_1 + t\boldsymbol{b}_2 = \boldsymbol{0} \Rightarrow \begin{pmatrix} s+2t \\ -s+t \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and since $s=t=0$ is the only solution, columns of $\boldsymbol{B}$ are linearly independent and span the subspace, hence are a basis for $\mathcal{S}$.

Ex 2.

$$\mathcal{S} = \left\{ \begin{pmatrix} a & a+d \\ a+d & d \end{pmatrix} | \, a,d \in \mathbb{R} \right\}, \mathcal{V} = \mathbb{R}^{2\times 2}$$

Ex 3. $\mathcal{S}$ is the space of skew-symmetric matrices ($\boldsymbol{A} \in \mathcal{S} \Rightarrow \boldsymbol{A}^T = -\boldsymbol{A}$), $\mathcal{V} = \mathbb{R}^{2\times 2}$

Ex 4. $\mathcal{S} = \{p(x) | p(0) = 0\}$, $\mathcal{V} = \mathcal{P}_2$.

Ex 5. $\mathcal{S} = \{p(x) | p(0) = 0, p(1) = 0\}$, $\mathcal{V} = \mathcal{P}_3$.

4. Find the coordinates of the vector $\boldsymbol{v}$ in the basis $\boldsymbol{B}$.

Ex 1.

$$\boldsymbol{B} = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}, \boldsymbol{v} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}, \mathcal{V} = \mathbb{R}^2$$

Solution. If $\boldsymbol{B}$ is a basis for $\mathcal{V} = \mathbb{R}^2$, then the vector $\boldsymbol{v}$ can be expressed as a linear combination of the basis vectors and that scalar coefficients are the coordinates of $\boldsymbol{v}$ in basis $\boldsymbol{B}$

$$\boldsymbol{v} = x_1\boldsymbol{b}_1 + x_2\boldsymbol{b}_2 = \boldsymbol{B}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \text{Solve} \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

to find the coordinates $x_1 = 2$, $x_2 = -1$. Check in Maxima.

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(%i5) B:matrix([3,-2],[1,2]);
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(%o5) $\begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$

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(%i6) v:matrix([8],[0]);
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(%o6) $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$

```
(%i7) linsolve_by_lu(B,v);

0 errors, 0 warnings
```

(%o7) $\left[\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \textbf{false} \right]$

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(%i8) x: first(%);
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(%o8) $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

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(%i9) B.x
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(%o9) $\begin{pmatrix} 8 \\ 0 \end{pmatrix}$

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(%i10)
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Ex 2.

$$\boldsymbol{B}=\begin{pmatrix} -2 & -1 \\ 4 & 1 \end{pmatrix}, \boldsymbol{v}=\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \mathcal{V}=\mathbb{R}^2$$

Ex 3.

$$\boldsymbol{B}=\begin{pmatrix} 1 & 3 & 1 \\ -1 & -1 & 0 \\ 2 & 1 & 2 \end{pmatrix}, \boldsymbol{v}=\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \mathcal{V}=\mathbb{R}^3$$

Ex 4.

$$\boldsymbol{B}=\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \boldsymbol{v}=\begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \end{pmatrix}, \mathcal{V}=\mathbb{R}^3$$

Ex 5.

$$\boldsymbol{B}=\begin{pmatrix} 1 & x-1 & x^2 \end{pmatrix}, \boldsymbol{v}=-2x^2+2x+3, \mathcal{V}=\mathcal{P}_2$$