Overview

- Rates of change arise in all the sciences
- Mathematics introduces appropriate abstractions for rates of change
- Features of the mathematical abstraction allows understanding of the phenomenon of interest, irrespective of the application domain
- Separable ODEs solved by integration
- Direction fields
- Software capabilities

(Textbook: 1.1-1.2, pp.2-26)

- What's the state of the solar system?

  Position vector of Sun, Mercury, Venus, Earth, ...: $\boldsymbol{r}_{\odot}(t), \boldsymbol{r}_{☿}(t), \boldsymbol{r}_{♀}(t), \boldsymbol{r}_{\oplus}(t)$

- What's the state of cesium decay?

  Number of cesium-137 isotopes: ${}^{137}\mathrm{Cs}(t)$

- What's the state of the US population?

  Number of people in the USA: $N(t)$

- What's the state of Congress?

  Gallup Congress Job Approval: $\mathrm{GALCOM}(t)$

- What's the state of the stock market?

  Dow Jones, NASDAQ, S&P: $\mathrm{DJIA}(t), \mathrm{NDX}(t), \mathrm{SPX}(t)$

- How do the planet positions change? Sun and planets: $\mathbf{r} = (\boldsymbol{r}_{\odot}, \boldsymbol{r}_{☿}, \boldsymbol{r}_{♀}, \boldsymbol{r}_{\oplus}, \ldots)$

$$\Delta \mathbf{r} = \mathbf{r}(t+\Delta t) - \mathbf{r}(t) = \int_t^{t+\Delta t} \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\,\mathrm{d}t = \int_t^{t+\Delta t} \mathbf{f}(t,\mathbf{r})\,\mathrm{d}t, \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \mathbf{f}(t,\mathbf{r})$$

- How does the number of ${}^{137}\mathrm{Cs}$ nuclei change? Use notation $n(t) = {}^{137}\mathrm{Cs}(t)$

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\lambda n$$

- How does the population change? Use notation $P(t)$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1 - \frac{P}{K}\right)$$

- How does the stock market change? $I'(t) = M(t, I, \omega)$

- All previous phenomena, from disparate disciplines conform to general form

$$y' = f(x, y)$$

and a solution $y(x)$ satisfies $y'(x) = f(x, y(x))$.

- Mathematical abstraction distills away specifics of a particular problem (y can be planet position, number of isotopes, approval rating, ...) on focuses on the essential

$$\text{Instantaneous rate of change} = \text{some function of current state}$$

- The two abstractions that arise are:
  - instantaneous rate of change y'
  - function of current state $f(x, y)$

Mathematical abstraction of rate of change

- Instantaneous rate of change

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$$

- Above definition is usually replaced by formal manipulations, e.g.,

$$y(x) = e^{\sin(x)} \Rightarrow y'(x) = e^{\sin(x)} \cos(x)$$

- Function of current state, $f(x, y)$, $f\colon \mathbb{R}^2 \to \mathbb{R}$

  Recall: for each (x, y), f returns a unique values $f(x, y)$

Benefits of mathematical abstraction

- Allows focus on issues common to *all* models, such as:
  - Can one always find a solution $y'(x) = f(x, y(x))$?
  - Are there any special properties $f(x, y)$ must have to allow a solution?
  - Is there only one solution to $y' = f(x, y)$?
  - Are there special forms of $f(x, y)$ that allow simple solutions?
  - What forms of $f(x, y)$ lead to complicated analytical solutions of $y' = f(x, y)$?
  - If an analytical solution of $y' = f(x, y)$ is difficult to find, are there alternative ways to find a solution?
  - What happens when the state is specified by multiple variables?

$$\boldsymbol{y}' = \boldsymbol{f}(x, \boldsymbol{y}), \boldsymbol{y} \in \mathbb{R}^n$$

- A particular case is $y' = f(x)$, i.e., there is no dependence of the slope on y

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \Rightarrow \mathrm{dy} = f(x)\,\mathrm{d}x$$

- The solution is found by integration

$$\int \mathrm{d}y = y(x) = \int f(x)\,\mathrm{d}x + C$$

Example: $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(x) \Rightarrow y(x) = -\cos(x) + C$

```
(%i4) ode2('diff(y,x)=sin(x),y,x);
```

(%o4) $y = \%c - \cos(x)$

- Another particular form is $f(x,y) = g(x)/h(y)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y) = \frac{g(x)}{h(y)} \Rightarrow h(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x$$

- Again, solution is found by integration

$$\int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

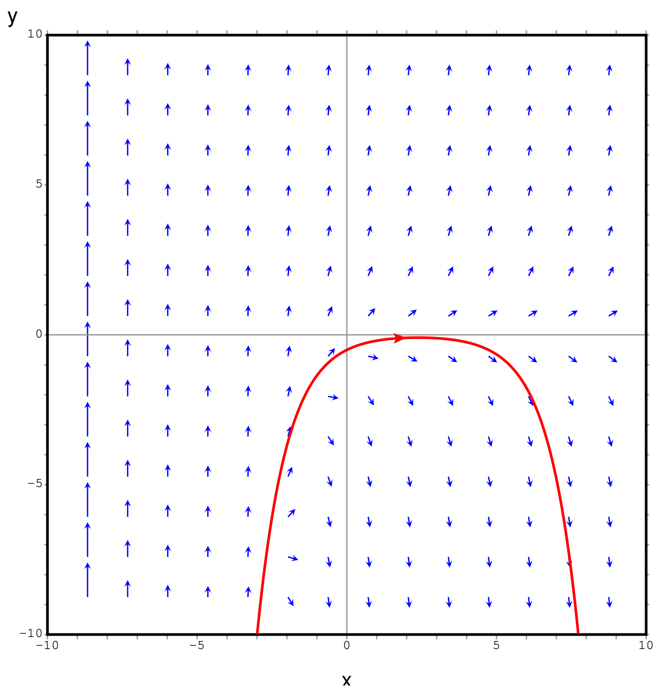
Example: $y' = \cos(x)/y \Rightarrow \int y\,\mathrm{d}y = \int \cos(x)\,\mathrm{d}x \Rightarrow \frac{1}{2}y^2 = \sin(x) + C$

```
(%i5) ode2('diff(y,x)=cos(x)/y,y,x);
```

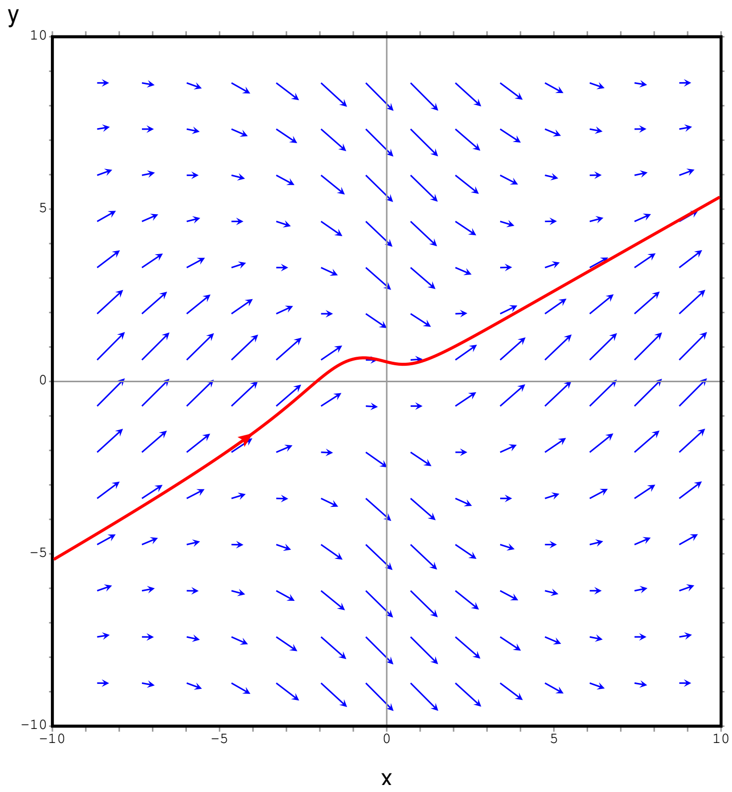
(%o5) $\frac{y^2}{2} = \sin(x) + \%c$

- In $y' = f(x, y)$, $f(x, y)$ is seen to specify the tangent to the curve $y(x)$
- Evaluation of $f(x, y)$ for $(x, y) \in \mathbb{R}^2$ leads to the definition of a *direction field*
- A plot of the direction field allows qualitative evaluation of a solution

```
(%i2) plotdf(exp(-x)+y,[trajectory_at,2,-0.1]);
```



```
(%i10) plotdf((x^2-y^2)/(1+x^2+y^2),[trajectory_at,-4,-1.5]);
```



- TeXmacs is an open-source program to produce mathematical documents
- TeXmacs allows inclusion of computational results from Maxima

```
(%i1) integrate(x,x);
```

(%o1) $$\frac{x^2}{2}$$

```
(%i2) integrate(x/(x^3+1),x);
```

(%o2) $$\frac{\log\left(x^2-x+1\right)}{6}+\frac{\arctan\left(\frac{2\,x-1}{\sqrt{3}}\right)}{\sqrt{3}}-\frac{\log\left(x+1\right)}{3}$$

```
(%i6) powerseries(sin(x),x,0);
```

(%o6) $$\sum_{i1=0}^{\infty}\frac{(-1)^{i1}\,x^{1+2\,i1}}{(1+2\,i1)!}$$

```
(%i7)
```

- Maxima is an open source descendent of Macsyma, a software package for symbolic computations

```
(%i6) integrate(exp(a*x)*sin(2*x)/2,x);
```

(%o6) $$\frac{e^{a\,x}\,\left(a\,\sin\left(2\,x\right)-2\,\cos\left(2\,x\right)\right)}{2\,\left(a^2+4\right)}$$

```
(%i7) diff(%,x);
```

(%o7) $$\frac{a\,e^{a\,x}\,\left(a\,\sin\left(2\,x\right)-2\,\cos\left(2\,x\right)\right)}{2\,\left(a^2+4\right)}+\frac{e^{a\,x}\,\left(2\,a\,\cos\left(2\,x\right)+4\,\sin\left(2\,x\right)\right)}{2\,\left(a^2+4\right)}$$

```
(%i8) ratsimp(%);
```

(%o8) $$\frac{e^{a\,x}\,\sin\left(2\,x\right)}{2}$$

```
(%i9)
```