Overview

- Linear first order differential equations
- Solution existence, uniqueness for first order differential equations
- Nonlinear first order differential equations: transformation to separable DEs

(Textbook: 2.1-2.2 , pp.30-52)

- *Implicit form* of a first order DE: $F(x, y, y') = 0$
- *Explicit form* of a first order DE: $y' = f(x, y)$
- The term *linear function* has two different meanings:
  - *In calculus*, $g(x)$ is a *linear function* if it is of the form $g(x) = ax + b$
  - *In linear algebra*, $g(x)$ is a *linear function* if $g(ax + by) = a\,g(x) + b\,g(y)$
- A bivariate function can be linear in one of its arguments, i.e., $f(x, y)$ is linear in y, if it is of the form $f(x, y) = a(x)\,y + b(x)$
- An explicit, first-order DE is said to be linear if the slope function is linear in the dependent variable, $y' = f(x, y) = a(x)\,y + b(x) \Rightarrow$

$$y' + p(x)\,y = q(x)$$

with $p(x) = -a(x)$, $q(x) = b(x)$

- The linear DE $y' + p(x)y = q(x)$, is said to be *homogeneous* if $q(x) = 0$
- A linear homogeneous DE always has $y = 0$ as a solution, known as *the trivial solution*. Any other solution, $y \neq 0$ is *nontrivial*.
- A homogeneous, linear first-order DE is separable. Formally, one finds:

$$y' + p(x)y = 0 \Rightarrow \frac{y'}{y} = -p \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\ln y = -p \Rightarrow \ln y = -\int p(x)\mathrm{d}x + \ln C$$

$$y(x) = Ce^{-\int p(x)\,\mathrm{d}x} = Ce^{-P(x)}, \text{with } P(x) = \int p(x)\,\mathrm{d}x \tag{1}$$

- "Formally" because appropriate mathematical conditions must be stated

Theorem. *$y(x) = Ce^{-\int p(x)\,\mathrm{d}x}$ is the general solution of $y'(x) + p(x)y = 0$ for $x \in (a, b)$, if $p(x)$ is continuous on (a, b).*

- Consider now

$$y' + p(x)\, y = q(x) \tag{2}$$

- Let $y_h(x)$ denote a non-trivial solution of $y' + p(x)\, y = 0$ (homogeneous)
- Method of variation of parameters: seek solution of (2) as $y(x) = u(x)\, y_h(x)$
- Compute $y' = u'\, y_h + u\, y_h'$, and replace in (2)

$$u'\, y_h + u\, y_h' + p(x)\, u(x)\, y_h(x) = q(x) \Rightarrow u' = \frac{q(x)}{y_h(x)} \Rightarrow u(x) = \int \frac{q(x)}{y_h(x)} \mathrm{d}x + C$$

- Solution of (2) is

$$y(x) = \left[\int \frac{q(x)}{y_h(x)} \mathrm{d}x + C \right] y_h(x).$$

Examples

Example: $y' + 2y = x^3 e^{-2x}$

Homogeneous: $y' + 2y = 0 \Rightarrow y_h(x) = e^{-2x}$

Variation of parameters: $y(x) = u(x)\, e^{-2x} \Rightarrow u' = x^3 \Rightarrow u(x) = \left(\frac{1}{4}x^4 + C\right)$

```
(%i13) ode2('diff(y,x)+2*y=0,y,x);
```

(%o13) $y = \%c\, e^{-2x}$

```
(%i12) ode2('diff(y,x)+2*y=x^3*exp(-2*x),y,x);
```

(%o12) $y = \left(\frac{x^4}{4} + \%c\right) e^{-2x}$

```
(%i13)
```

Theorem. *If* p, q *are continuous on* (a, b), *and* y_h *satisfies*

$$y_h' + p(x) y_h = 0,$$

then:

a) *the general solution of* $y' + p(x)\, y = q(x)$ *on* (a, b) *is*

$$y(x) = y_h(x)\left(c + \int \frac{q(x)}{y_h(x)}\, \mathrm{d}x\right);$$

b) *if* $x_0 \in (a, b)$, $y_0 \in \mathbb{R}$, *the unique solution of* $y' + p(x)\, y = q(x)$, $y(x_0) = y_0$ *is*

$$y(x) = y_h(x)\left(\frac{y_0}{y_h(x_0)} + \int_{x_0}^{x} \frac{q(t)}{y_h(t)}\, \mathrm{d}t\right).$$

Nonlinear differential equations

- Separable nonlinear equations can be solved by integration

$$y' = \frac{g(x)}{h(y)} \Rightarrow h(y)\,\mathrm{d}y = g(x)\,\mathrm{d}x \Rightarrow \int h(y)\,\mathrm{d}y = \int g(x)\,\mathrm{d}x$$

- Some non-separable, nonlinear equations can be reduced to separable equations by transformation of the dependent variable

  Example: Bernoulli equation

$$y' + p(x)\,y = q(x)\,y^r, r \in \mathbb{R}\backslash\{0, 1\} \tag{3}$$

```
(%i15) plotdf(y*(1+x*y),[trajectory_at,-2,-0.2]);
```