Overview

- More on transformation of nonlinear differential equations to separable DEs:
  - Bernoulli example
  - Homogeneous equations

(Textbook: 2.4)

Transformation of Bernoulli equations

- A Bernoulli equation, $y' + p(x)\, y = q(x)\, y^r, r \in \mathbb{R} \backslash \{0, 1\}$ can be transformed into a separable equation by variation of parameters
  - Homogeneous equation $y' + p(x)\, y = 0$ has solution $y_h(x)$
  - Seek solution of form $y = u y_h$ (variation of parameters)

$$(u y_h)' + p u y_h = q\,(u y_h)^r \Rightarrow y_h u' + u(y_h + p y_h) = q\,(u y_h)^r \Rightarrow u' = q u^r\, y_h^{r-1}$$

  - The last equation is separable and integration leads to

$$\int \frac{u'}{u^r}\,\mathrm{d}x = \int q(x)\, y_h^{r-1}\,\mathrm{d}x \Rightarrow \frac{1}{1-r}\,u^{1-r} = \int q(x)\, y_h^{r-1}\,\mathrm{d}x$$

$$u(x) = \left[(1-r)\int q(x)\, y_h^{r-1}\,\mathrm{d}x\right]^{\frac{1}{1-r}}.$$

Solve $y' - y = \sin(x)\, y^3$. Steps:

- Homogeneous equation solution: $y' - y = 0 \Rightarrow y(x) = e^x$
- Variation of parameters: $y = u e^x \Rightarrow u' e^x = \sin(x) u^3 e^{3x} \Rightarrow u' u^{-3} = \sin(x)\, e^{2x} \Rightarrow$ $\int \frac{u'}{u^3} \mathrm{d}x = -\frac{1}{2u^2} = \int \sin(x)\, e^{2x} + C = \frac{\mathrm{e}^{2x} (2 \sin(x) - \cos(x))}{5} + C.$

$$u(x) = \left[c - \frac{2}{5} e^{2x} (2 \sin(x) - \cos(x)) \right]^{-1/2}$$

- Form solution: $y = u e^x = e^x \left[c - \frac{2}{5} e^{2x} (2 \sin(x) - \cos(x)) \right]^{-1/2}$

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(%i7) ode2('diff(y,x)-y=sin(x)*y^3,y,x);
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(%o7) $y = \frac{\mathrm{e}^x}{\sqrt{\%c - \frac{2\, \mathrm{e}^{2x} (2 \sin(x) - \cos(x))}{5}}}$

$$y' = y + \sin(x)\, y^3 = y(1 + \sin(x) y^2)$$

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(%i8) plotdf(y*(1+sin(x)*y^2),[trajectory_at,-2,-0.2]);
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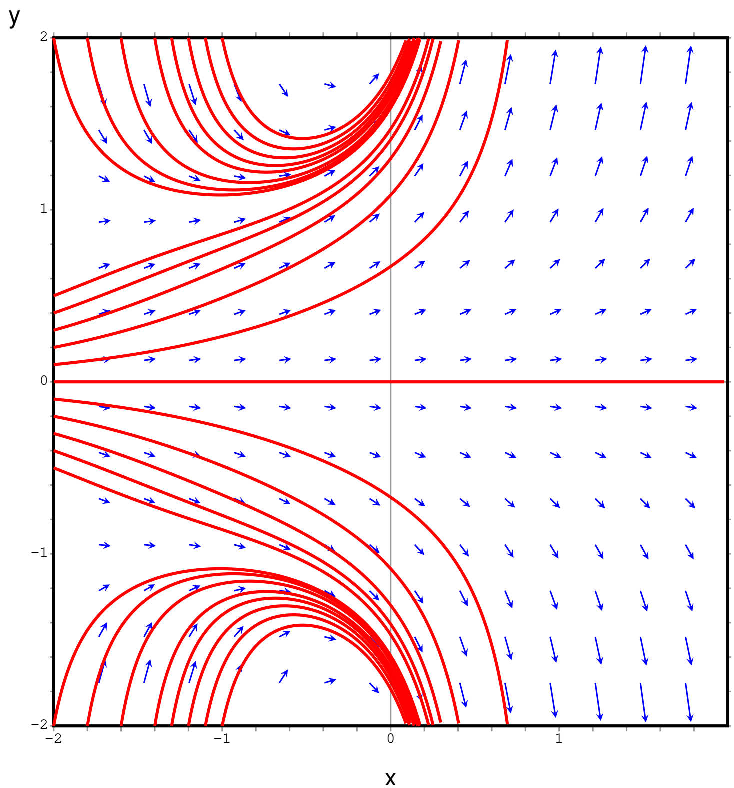

Definition. *The differential equation* $y' = f(x, y)$ *is said to be homogeneous if it can be written as* $y' = g(y/x)$.

Homogeneous equations are separable through the transformation $u = y/x$,

$$y' = (xu)' = xu' + u = g(u) \Rightarrow \frac{u'}{g(u) - u} = \frac{1}{x} \Rightarrow \ln x = \int \frac{u'}{g(u) - u} \mathrm{d}x + C$$

Example: $y' = \frac{y}{x} + e^{y/x}$, $u = \frac{y}{x} \Rightarrow xu' + u = u + e^u \Rightarrow u' e^{-u} = x^{-1}$, with solution $e^{-u} = c - \ln x \Rightarrow u = -\ln(c - \ln(x))$, leading to $y = xu = -x\ln(c - \ln x)$

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(%i15) y: -x*log(c-log(x)); fullratsimp(diff(y,x)-y/x-exp(y/x));
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(%o15) $-x \log(c - \log(x))$
(%o16) 0

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(%i17)
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