Overview

- Exact solutions to first-order differential equations
- Examples

Rewrite a first-order DE as a relationship between infinitesimal increments

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \Leftrightarrow M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0. \text{ (exact differential)}$$

Compare with $\mathrm{d}F(x, y) = F_x\,\mathrm{d}x + F_y\,\mathrm{d}y$.

Theorem. *If $F\colon \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives F_x, F_y, then*

$$F(x, y) = c,$$

is an implicit solution of $F_x(x, y)\,\mathrm{d}x + F_y(x, y)\,\mathrm{d}y = 0$. ($F_x \equiv M, F_y \equiv N$)

Theorem. *If $M, N\colon \mathbb{R}^2 \to \mathbb{R}$, are continuous with continuous partial derivatives in some open rectangle R, then $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y$ is an exact differential on R if and only if $M_y(x, y) = N_x(x, y)$ in R.*

Example 1

$$F(x,y) = x^4y^3 + x^2y^5 + 2xy = c$$

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(%i11) F: x^4*y^3+x^2*y^5+2*x*y$
(%i12) diff(F,x);
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(%o12) $2\,x\,y^5 + 4\,x^3\,y^3 + 2\,y$

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(%i13) diff(F,y);
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(%o13) $5\,x^2\,y^4 + 3\,x^4\,y^2 + 2\,x$

$$(2\,x\,y^5 + 4\,x^3\,y^3 + 2\,y)\mathrm{d}x + (5\,x^2\,y^4 + 3\,x^4\,y^2 + 2\,x)\mathrm{d}y = 0 \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2\,x\,y^5 + 4\,x^3\,y^3 + 2\,y}{5\,x^2\,y^4 + 3\,x^4\,y^2 + 2\,x}, \frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{5\,x^2\,y^4 + 3\,x^4\,y^2 + 2\,x}{2\,x\,y^5 + 4\,x^3\,y^3 + 2\,y}.$$

Example 2

Check if the following differential form is exact

$$3x^2 y\,\mathrm{d}x + 4x^3\,\mathrm{d}y = 0.$$

Denote: $M(x, y) = 3x^2 y, N(x, y) = 4x^3$. Compute

$$M_y = 3x^2, N_x = 12x^2 \Rightarrow M_y \neq N_x \Rightarrow \text{not exact}.$$

Example 3 - solution 1

Consider $M(x,y)=4x^3y^3+3x^2, N(x,y)=3x^4y^2+6y^2$. Find $F(x,y)$ such that $F_x=M$, $F_y=N$ if $M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0$.

1. Integrate $F_x=M$ w.r.t. x

$$\int F_x\,\mathrm{d}x=\int M\,\mathrm{d}x \Rightarrow F-f(y)=\int(4x^3y^3+3x^2)\mathrm{d}x=x^4y^3+x^2.$$

2. Replace $F=f(y)+x^4y^3+x^3$, in $F_y=N$

$$f'+3x^4y^2=3x^4y^2+6y^2.$$

3. Integrate to find f

$$f=2y^3\Rightarrow F=x^4y^3+x^3+2y^3.$$