Overview

- Motivation: systems described by multiple state parameters
- Basic concepts: scalars, vectors, matrices
- Sets: finite, countable, countably infinite, uncountably infinite
- Algebraic structures: groups, fields, field of real numbers
- Functions: general definition, vectors as functions defined on a finite set

- Up to now, systems described by $y: \mathbb{R} \to \mathbb{R}$, $y' = f(x, y)$ have been considered
- Most systems require more than one parameter to establish their state, e.g.,
  - Position in 3D-space: $r(t) = \begin{pmatrix} x(t) & y(t) & z(t) \end{pmatrix} \in \mathbb{R}^3$, $r: \mathbb{R} \to \mathbb{R}^3$;
  - Stock market prices: $p: \mathbb{R} \to \mathbb{R}^n$, n the number of listed stocks,
  - Patient health monitoring: $y = \begin{pmatrix} \text{temperature} & \text{heart rate} & \dots \end{pmatrix}$
- Change in system whose state is given by n parameters that each depend on a single independent variable x

$$y' = f(x, y), y: \mathbb{R} \to \mathbb{R}^n, f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \tag{1}$$

- The mathematical issue that arises is to establish a framework to work with multi-dimensional quantitites, such as $y(x)$ from (1)

- A *scalar* is a quantity described completely by its magnitude. Such quantities are described mathematically by a single number such as $n \in \mathbb{N}$ or $\alpha \in \mathbb{R}$
- Informally a vector is a quantity that besides magnitude also requires the specification of a direction. This usage arises from common examples in physics, e.g., the position vector of a point $\vec{r} = x\vec{i} + y\vec{i} + z\vec{k}$. Consider for now a vector to be a grouping of multiple scalars, e.g.,

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m.$$

  A mathematically precise definition is given later.
- Just as scalars are grouped into vectors, n vectors are grouped into a *matrix*

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \dots & \boldsymbol{a}_n \end{pmatrix} \in \mathbb{R}^{m \times n}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_n \in \mathbb{R}^m.$$

- The informal introduction of vectors, matrices can lead to contradictions. A consistent mathematical framework is required. First step: *naïve set theory*.

- In naïve set theory[1], a *set* is a collection of *objects*, e.g., $A = \{a, 2, Z\}$, and a fundamental binary relation between objects and sets is introduced, denoted by $\in$, the *member of relation*, such as $2 \in A$. The negation of this relation is $\notin$, the *not member of relation*, e.g., $3 \notin A$.

- Some sets are *finite*, e.g., $A = \{a, 2, Z\}, B = \{1, 2, \ldots, n\}$

- Other sets are infinite, but countable: $\mathbb{N} = \{0, 1, 2, \ldots\}$,$\mathbb{N}^* = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q} = \{a/b | a \in \mathbb{Z}, b \in \mathbb{N}^*\}$

- Yet other sets are infinite and not-countable: $\mathbb{R}$, $\mathbb{C}$

1. The resulting set theory is called "naïve" because it can lead to paradoxes, e.g., the Russell paradox: "What is the set of all sets that are not members of themselves". Mathematically: define $R = \{x | x \notin x\}$. Then the paradox is that $R \in R \Leftrightarrow R \notin R$, i.e. if R is a member of R then it is not a member and if R is not a member of R then it is a member. Such paradoxes led to the formulation of axiomatic set theory.

- After introducing objects and collections of objects in set theory, the next step is to formally define operations with objects. *Algebra* defines many such formal structures. Of particular interest here are groups and fields.

- A *group* is the algebraic structure $(\mathcal{S}, \oplus)$ with $\mathcal{S}$ a set, $\oplus$ a binary operation with properties for $\forall a, b, c \in \mathcal{S}$:

  Closure. $a \oplus b \in \mathcal{S}$.

  Associativity. $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

  Identity element. $\exists e \in \mathcal{S}$, such that $a \oplus e = e \oplus a = a$

  Inverse element. $\exists (-a)$ such that $a \oplus (-a) = e$

- In addition to the above properties, a *commutative group* also satisfies:

  Commutativity. $a + b = b + a$.

- A field is an algebraic structure $(\mathcal{S}, \oplus, \otimes)$ with commutative group properties for $\oplus$ and $\otimes$ (with an exception) and distributivity of $\otimes$ over $\oplus$, $\forall a, b, c \in \mathcal{S}$

$a \oplus b \in \mathcal{S}$	$a \otimes b \in \mathcal{S}$
$a \oplus (b \oplus c) = (a \oplus b) \oplus c$	$a \otimes (b \otimes c) = (a \otimes b) \otimes c$
$\exists 0 \in \mathcal{S}$, $a \oplus 0 = 0 \oplus a = a$	$\exists 1 \in \mathcal{S}$, $a \otimes 1 = 1 \otimes a = a$
$\exists (-a), a + (-a) = 0$	if $a \neq 0$, $\exists a^{-1}$, $a \otimes a^{-1} = 1$
$a \oplus b = b \oplus a$	$a \otimes b = b \otimes a$

  Distributivity. $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c)$

- The real numbers with addition and multiplication form a field $(\mathbb{R}, +, \cdot)$
- The complex numbers with addition and multiplication form a field $(\mathbb{C}, +, \cdot)$

- A *function* f is a relation between two sets $\mathcal{D}, \mathcal{C}$ that associates to every element of $\mathcal{D}$ (the *domain*) a single element from $\mathcal{C}$ (the *codomain*)
- Notations: $f: \mathcal{D} \to \mathcal{C}$, $\mathcal{D} \xrightarrow{f} \mathcal{C}$, $x \in \mathcal{D}$, $f(x) \in \mathcal{C}$
- An important observation is that vectors can be interpreted as functions:
  - Consider $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} = (\ x_1 \ \ \dots \ \ x_n\ )^T$ (here the ^{T} superscript indicates transposition of the row vector into a column vector).
  - Now consider the function $x: \{1, \dots, n\} \to \mathbb{R}$ defined as $x(i) = x_i$

The above observations presages that many of the results for the algebra of vectors within $\mathbb{R}^n$ have relevance and generalization for other types of vectors. In particular, functions $y: \mathbb{R} \to \mathbb{R}$ can be interpreted as vectors within some set $\mathcal{C}$ (different from $\mathbb{R}^n$)