Overview

- Algebraic structure of a vector space, $(\mathbb{R}^n, \mathbb{R}, +, \cdot)$
- Linear combination, matrix-vector product
- Linear independence, linear dependence
- Range and null space

- After groups and fields, an additional algebraic structure of particular relevance to differential equations is now introduced, that of a vector space
- A *vector space* $(\mathcal{V}, (\mathcal{S}, +, \cdot), \oplus, \odot)$ is formed by a set of vectors $\mathcal{V}$, a set of scalars $(\mathcal{S}, +, \cdot)$ with a field structure, the operation of *vector addition* $\oplus$, and the operation of *multiplication of a vector by a scalar* $\odot$, with the properties of
  - a commutative group for $(\mathcal{V}, \oplus)$: $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$, $\forall \alpha, \beta \in \mathcal{S}$

$$\boldsymbol{u} \oplus \boldsymbol{v} \in \mathcal{V} \qquad \boldsymbol{u} \oplus (\boldsymbol{v} \oplus \boldsymbol{w}) = (\boldsymbol{u} \oplus \boldsymbol{v}) \oplus \boldsymbol{w}$$

$$\exists \mathbf{0} \in \mathcal{V},\ \boldsymbol{u} \oplus \mathbf{0} = \mathbf{0} \oplus \boldsymbol{u} = \boldsymbol{u} \qquad \exists (-\boldsymbol{u}),\ \boldsymbol{u} \oplus (-\boldsymbol{u}) = 0$$

$$\boldsymbol{u} \oplus \boldsymbol{v} = \boldsymbol{v} \oplus \boldsymbol{u}$$

  - closure with respect to multiplication by a scalar: $\alpha \boldsymbol{u} \in \mathcal{V}$
  - Distributivity properties:

$$\alpha \odot (\boldsymbol{u} \oplus \boldsymbol{v}) = \alpha \odot \boldsymbol{u} \oplus \alpha \odot \boldsymbol{v} \qquad (\alpha + \beta) \odot \boldsymbol{u} = \alpha \odot \boldsymbol{u} \oplus \beta \odot \boldsymbol{u}$$

$$\alpha \odot (\beta \odot \boldsymbol{u}) = (\alpha \cdot \beta) \boldsymbol{u}$$

  - Scalar identity element: $1 \odot \boldsymbol{u} = \boldsymbol{u}$

- $(\mathcal{P}_n, +, \mathbb{R}, \cdot)$ is the *vector space of polynomials* of degree at most n over the reals. For $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}_n$ $\alpha \in \mathbb{R}$, the operations $+, \cdot$ are defined as

$$\boldsymbol{p} = a_0 + a_1 x + \cdots + a_n x^n, \qquad \boldsymbol{q} = b_0 + b_1 x + \ldots + b_n x^n$$
$$\boldsymbol{p} + \boldsymbol{q} = a_0 + b_0 + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n,$$
$$\alpha \boldsymbol{p} = \alpha a_0 + \alpha a_1 x + \cdots + \alpha a_n x^n,$$

  with $a_n, \ldots a_0 \in \mathbb{R}$, $b_n, \ldots b_0 \in \mathbb{R}$. Note that the variable x is irrelevant to the specification of a particular polynomial by its coefficients, such that it is convenient to identify $\boldsymbol{p} \equiv (a_0, a_1, \ldots, a_n)$ and the operations become

$$\boldsymbol{p} + \boldsymbol{q} = (a_0 + b_0, a_1 + b_1, \ldots, a_n + b_n) \quad \alpha \boldsymbol{p} = (\alpha a_0, \ldots, \alpha a_n)$$

- $(\mathbb{R}, \oplus, \mathbb{R}, \cdot)$ with $a, b \in \mathcal{V} = \mathbb{R}$, $a \oplus b = a^b$ is not a vector space, since $a^b \neq b^a$ in general. $(\mathcal{P}_n, +, \mathbb{N}, \cdot)$ is not a vector space since $\mathbb{N}$ is not a field.

- The *Euclidean vector space* $(\mathbb{R}^n, \mathbb{R}, +, \cdot)$ with operations for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, \boldsymbol{x} + \boldsymbol{y} = \begin{pmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{pmatrix}, \alpha \boldsymbol{x} = \begin{pmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

  has many applications:

  – position vector of a point in 3D, $\vec{r} = x\,\vec{i} + y\,\vec{i} + z\,\vec{k}$

  – polynomials, $\boldsymbol{p} = a_0 + a_1 x + \cdots + a_n x^n$, $\boldsymbol{p} = (a_0, a_1, \ldots, a_n)$

  – trigonometric sums, $\boldsymbol{f} = a_0 + a_1 \cos t + b_1 \sin t + \cdots + a_n \cos t + b_n \sin t$, $\boldsymbol{f} \equiv (a_0, a_1, b_1, \ldots, a_n, b_n) \in \mathbb{R}^{2n+1}$

  – exponential sums, $\boldsymbol{f} = a_0 + a_1 e^t + \cdots + a_n e^{nt}$

In vector space $(\mathcal{V},+,\mathcal{S},\cdot)$, from $\boldsymbol{a}_1, \boldsymbol{b}_2 \in \mathcal{V}$, $x_1, x_2 \in \mathcal{S}$, obtain a new vector $\boldsymbol{w}$

$$\boldsymbol{b} = x_1 \boldsymbol{a}_1 + x_2\, \boldsymbol{a}_2 \tag{1}$$

- Question: can all vectors within $\mathcal{V}$ be obtained this way?

  Answer: in general, no since in $(\mathbb{R}^3,+,\mathbb{R},\cdot)$, $\boldsymbol{b} = (0,0,1)$ cannot be obtained from $\boldsymbol{a}_1 = (1,0,0)$, $\boldsymbol{a}_2 = (0,1,0)$.

- Question: can one somehow generalize (1) so as to describe all vectors in $\mathcal{V}$?

  Answer: yes, through the following

Definition. *In vector space* $(\mathcal{V},+,\mathcal{S},\cdot)$

$$\boldsymbol{b} = x_1\, \boldsymbol{a}_1 + x_2\, \boldsymbol{a}_2 + \cdots + x_n\, \boldsymbol{a}_n$$

is the linear combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathcal{V}$ *with coefficients* $x_1, \ldots, x_n \in \mathcal{S}$.

Linear combinations have very many applications. It is convenient to define notation to efficiently carry out linear combinations, especially for the widely used Euclidean space $(\mathbb{R}^m, +, \mathbb{R}, \cdot)$ (we use m for number of components, n

1. Group the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ together to define a *matrix*, group the scalar coefficients $x_1, x_2, \ldots, x_n \in \mathbb{R}$ together into a vector $\boldsymbol{x} \in \mathbb{R}^n$

$$\boldsymbol{A} = \left(\begin{array}{ccc} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{array} \right) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

2. Define matrix-vector multiplication to carry out the linear combination

$$\boldsymbol{A}\boldsymbol{x} = x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \cdots + x_n \boldsymbol{a}_n$$

Note: m denotes the number of components in a vector, n the number of vectors in the linear combination.

- Denote set of real-component matrices with m rows and n columns by $\mathbb{R}^{m\times n}$
- Notation:
  - $\boldsymbol{A}, \boldsymbol{B}, \dots$ are matrices, $\boldsymbol{x}, \boldsymbol{b}$ are vectors, u, v are scalars
  - Column vectors of $\boldsymbol{B} \in \mathbb{R}^{m\times n}$ are $\boldsymbol{b}_1, \dots, \boldsymbol{b}_n$
  - Components of $\boldsymbol{a}_j \in \mathbb{R}^m$ are $a_{1j}, \dots, a_{mj}$
- Matrix-vector product can also be computed by "row over columns rule"

$$\boldsymbol{A}\boldsymbol{x} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 a_{11} + x_2 a_{12} + \dots + x_n a_{1n} \\ x_1 a_{21} + x_2 a_{22} + \dots + x_n a_{2n} \\ \vdots \\ x_1 a_{m1} + x_2 a_{m2} + \dots + x_n a_{mn} \end{pmatrix}$$

Having defined $\boldsymbol{A}\boldsymbol{x}$ as a concise way of defining the linear combination of n vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathbb{R}^m$ with scalar coefficients $x_1, \ldots, x_n \in \mathbb{R}$, consider now the original question: can all vectors within a vector space $\mathcal{V}$ be reached by linear combination of some choice of vectors in $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathcal{V}$?

Definition. *The vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathcal{V}$ are said to be linearly dependent if there exists a solution $\boldsymbol{x} \neq \boldsymbol{0}$ to the equation $\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$.*

- In the vector space $(\mathbb{R}^2, +, \mathbb{R}, \cdot)$, $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ are linearly dependent. The solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ are $x_1 = -2\alpha$, $x_2 = \alpha$, for any $\alpha \in \mathbb{R}$
- In the vector space $(\mathcal{P}_1, +, \mathbb{R}, \cdot)$, $\boldsymbol{A} = \begin{pmatrix} 1 & t & 1-t \end{pmatrix}$ are linearly dependent since $\boldsymbol{a}_3 = 1 - t = 1 \cdot \boldsymbol{a}_1 - 2 \cdot \boldsymbol{a}_2 \Rightarrow 1 \cdot \boldsymbol{a}_1 - 2 \cdot \boldsymbol{a}_2 - \boldsymbol{a}_3 = \boldsymbol{0}$, and $\boldsymbol{x} = (1, -2, -1)$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ with $\boldsymbol{x} \neq \boldsymbol{0}$.[1]

1. Note: the matrix notation has been extended to allow for arbitrary vectors (in this case polynomials) to be placed in columns. This is a very useful convention for computational applications.

Definition. *The vectors* $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in \mathcal{V}$ *are said to be* *linearly independent* *if* $\boldsymbol{x} = \mathbf{0}$ *is the only solution to the equation* $\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{a}_1 + \cdots + x_n\boldsymbol{a}_n = \mathbf{0}$.

- In the vector space $(\mathbb{R}^2, +, \mathbb{R}, \cdot)$, $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent. The only solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ is $x_1 = 0$, $x_2 = 0$
- In the vector space $(\mathcal{P}_1, +, \mathbb{R}, \cdot)$, $\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \boldsymbol{t} \end{pmatrix}$ are linearly independent since

$$\boldsymbol{A}\boldsymbol{x} = x_1 \cdot \mathbf{1} + x_2 \cdot \boldsymbol{t} = \mathbf{0} \tag{2}$$

  is an equality between vectors that are in this case the polynomials $\mathbf{1}, \boldsymbol{t} \in \mathcal{P}_1$, and (2) must be satisfied for all t. Choosing $t = 0$ implies $x_1 = 0$, and choosing $t = 1$ subsequently implies $x_2 = 0$

- Linear dependence of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ indicates that some vectors are redundant, they can be expressed as linear combinations of other vectors in the set.
- Linear independence of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ indicates that there are no redundant vectors in the set, but it is not yet established that all vectors in the $\mathcal{V}$ can be reached by linear combination of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.

Definition. *The range of $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{pmatrix}$ (also known as the span of vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$) is the set*

$$\mathcal{R}(\boldsymbol{A}) = \{\boldsymbol{b} : \exists \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} = \boldsymbol{A}\boldsymbol{x}\}.$$

The range or the span is the set of all vectors reachable by linear combination of columns of $\boldsymbol{A}$.

Also introduce a definition to characterize whether a set of vectors (represented as columns of a matrix) are linearly independent or not.

Definition. *The null space of* $\boldsymbol{A} = \left(\begin{array}{ccc} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{array} \right)$ *(also known as the kernel of* $\boldsymbol{A}$*) is the set*

$$\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \colon \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}.$$

If the null space only contains $\boldsymbol{0}$, $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{0}\}$, then the vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, $\boldsymbol{A} = \left(\begin{array}{ccc} \boldsymbol{a}_1 & \ldots & \boldsymbol{a}_n \end{array} \right)$ are linearly independent.

- In vector space $(\mathbb{R}^2, +, \mathbb{R}, \cdot)$, the null space of $\boldsymbol{A} = \left(\begin{array}{cc} \boldsymbol{a}_1 & \boldsymbol{a}_2 \end{array} \right) = \left(\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array} \right)$ is

$$\mathcal{N}(\boldsymbol{A}) = \left\{ \alpha \left(\begin{array}{c} -2 \\ 1 \end{array} \right) \colon \alpha \in \mathbb{R} \right\}, \boldsymbol{A} \left(\begin{array}{c} -2\alpha \\ \alpha \end{array} \right) = \left(\begin{array}{cc} 1 & 2 \\ 2 & 4 \end{array} \right) \left(\begin{array}{c} -2\alpha \\ \alpha \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$$