Overview

- Basis, dimension
- Norm, scalar product, orthogonality
- Change of basis in $\mathbb{R}^m$, $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{I}\boldsymbol{b}$, and $\mathcal{P}_n$, $\boldsymbol{B}(t)\,\boldsymbol{c}=\boldsymbol{M}(t)\,\boldsymbol{a}$

- Ask what is the minimal set of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \in \mathcal{V}$ needed to reach all other vectors in $\mathcal{V}$. This leads to the following:

Definition. *A basis for a vector space $(\mathcal{V}, +, \mathcal{S}, \cdot)$ is a set of vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots \in \mathcal{V}$ that are linearly independent and whose range is the entire vector space, $\mathcal{R}(\boldsymbol{A}) = \mathcal{V}$, with $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots \end{pmatrix}$*

Example 1. The column vectors of the identity matrix $\boldsymbol{I} \in \mathbb{R}^{m \times m}$ are a basis for $\mathbb{R}^m$

$$\boldsymbol{I} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \ldots & \boldsymbol{e}_m \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

Example 2. $\{1, t, t^2, \ldots, t^n\}$ is a basis for $\mathcal{P}_n$, polynomials up to degree n.

- The notion of a basis allows a precise definition of dimension, i.e., "the size of a vector space".

Definition. *Two sets $\mathcal{A}, \mathcal{B}$ are said to have the same cardinality (intuitively, the same size), denoted as $|\mathcal{A}| = |\mathcal{B}|$ if there is some one-to-one function (a bijection) between $\mathcal{A}$ and $\mathcal{B}$.*

Definition. *A set of the same cardinality as the $\{1, 2, \ldots, n\}$ subset of the naturals $\mathbb{N}$ is said to be a finite set.*

Definition. *The dimension of a vector space $(\mathcal{V}, +, \mathcal{S}, \cdot)$ is the cardinality of a basis $\boldsymbol{A} = (\ \boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \ldots\ )$ of $\mathcal{V}$.*

Note that the above sequence of definitions allows for definition of infinite-dimensional vector spaces, whose definition gets to be a bit technical, i.e., an infinite set is one that has a subset of cardinality n, for any $n \in \mathbb{N}$.

- Vectors were introduced for quantities with both magnitude and direction.
- The norm function $\| \,\| : \mathcal{V} \to \mathbb{R}_+$ extracts the magnitude of a vector

Definition. *A function $\| \,\| : \mathcal{V} \to \mathbb{R}_+$ is a norm on vector space $(\mathcal{V}, +, \mathcal{S}, \cdot)$ if for any vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ and any scalar $\alpha \in \mathcal{S}$*

a) $\|\boldsymbol{u}\| \geqslant 0$, *and* $\|\boldsymbol{u}\| = 0$ *if and only if* $\boldsymbol{u} = \boldsymbol{0}$ *(only the zero vector has zero norm)*

b) $\|\alpha \boldsymbol{u}\| = |\alpha| \, \|\boldsymbol{u}\|$ *(vectors can be scaled)*

c) $\|\boldsymbol{u} + \boldsymbol{v}\| \leqslant \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$ *(triangle inequality)*

Example. In the Euclidean vector space $(\mathbb{R}^m, +, \mathbb{R}, \cdot)$, $\|\boldsymbol{v}\| = \left(\sum_{i=1}^{m} v_i^2\right)^{1/2}$ is a norm (the Euclidean norm or the 2-norm).

- The norm can be used to measure magnitude, but another tool is needed to measure direction, such as the relative orientation between two vectors.

Definition. *A function* $(\,,\,) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ *is a scalar product over the vector space* $(\mathcal{V}, +, \mathbb{R}, \cdot)$ *if* $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathcal{V}$, $\forall \alpha, \beta \in \mathcal{S}$:

a) $(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{v}, \boldsymbol{u})$ *(symmetry of relative orientation)*

b) $(\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{w}) = \alpha (\boldsymbol{u}, \boldsymbol{w}) + \beta (\boldsymbol{v}, \boldsymbol{w})$ *(linearity in first argument)*

c) $(\boldsymbol{u}, \boldsymbol{u}) > 0$ *if* $\boldsymbol{u} \neq \boldsymbol{0}$.

Note that a scalar product can be used to define a norm through $\|\boldsymbol{u}\| = (\boldsymbol{u}, \boldsymbol{u})^{1/2}$.

Example. In $(\mathbb{R}^m, +, \mathbb{R}, \cdot)$, $(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i v_i$ is a scalar product. It can be computed through matrix multiplication as $(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{u}^T \boldsymbol{v}$.

- One special relative orientation between vectors is of great interest in linear algebra because it implies linear independence.

Definition. *Vectors $\boldsymbol{u}, \boldsymbol{v}$ are said to be orthogonal if their scalar product is null, $(\boldsymbol{u}, \boldsymbol{v}) = 0$.*

Orthogonal vectors are linearly independent. Consider $\alpha \boldsymbol{u} + \beta \boldsymbol{v} = \mathbf{0}$ and $(\boldsymbol{u}, \boldsymbol{v}) = 0$.

- Take the scalar product of $\alpha \boldsymbol{u} + \beta \boldsymbol{v}$ and $\boldsymbol{u}$: $(\alpha \boldsymbol{u} + \beta \boldsymbol{v}, \boldsymbol{u}) = \alpha(\boldsymbol{u}, \boldsymbol{u}) + \beta(\boldsymbol{v}, \boldsymbol{u}) = \alpha(\boldsymbol{u}, \boldsymbol{u}) + \beta(\boldsymbol{u}, \boldsymbol{v}) = \alpha(\boldsymbol{u}, \boldsymbol{u}) = (\mathbf{0}, \boldsymbol{u}) = 0$. Since $(\boldsymbol{u}, \boldsymbol{u}) > 0$ it results that $\alpha = 0$
- Repeat for scalar product of $\alpha \boldsymbol{u} + \beta \boldsymbol{v}$ and $\boldsymbol{v}$ to find that $\beta = 0$
- Since $\alpha \boldsymbol{u} + \beta \boldsymbol{v} = \mathbf{0}$ implies both $\alpha = 0$ and $\beta = 0$, $\boldsymbol{u}, \boldsymbol{v}$ are linearly independent.

- The above framework can be extended to discuss relative orientation and orthogonality (and hence, linear independence) of functions
- $(\mathcal{C}_{[a,b]}, +, \mathbb{R}, \cdot)$ is the vector space of continuous, real-valued functions defined on $[a, b]$, $\boldsymbol{f} \in \mathcal{C}_{[a,b]} \Rightarrow f\colon [a,b] \to \mathbb{R}$
- $(\boldsymbol{f}, \boldsymbol{g}) = \int_a^b f(t)\, g(t)\, \mathrm{d}t$ is a scalar product for $(\mathcal{C}_{[a,b]}, +, \mathbb{R}, \cdot)$
- If $(\boldsymbol{f}, \boldsymbol{g}) = 0$ the functions $\boldsymbol{f}, \boldsymbol{g} \in \mathcal{C}_{[a,b]}$ are said to be orthogonal and they are linearly independent

Example. $\sin t, \cos t$ are linearly independent in $(\mathcal{C}_{[0,2\pi]}, +, \mathbb{R}, \cdot)$ since

$$(\sin t, \cos t) = \int_0^{2\pi} \sin t \cos t \, \mathrm{d}t = \frac{1}{2}\int_0^{2\pi} \sin 2t \, \mathrm{d}t = \frac{-1}{4}[\cos 2t]_0^{2\pi} = 0.$$

- A vector from the Euclidean vector space $(\mathbb{R}^m, +, \mathbb{R}, \cdot)$ is typically given in terms of its components with respect to the $\boldsymbol{I} = (\boldsymbol{e}_1 \ \ \boldsymbol{e}_2 \ \ \dots \ \ \boldsymbol{e}_m)$ basis

$$\boldsymbol{b} = b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + \cdots + b_m\boldsymbol{e}_m = \boldsymbol{I}\boldsymbol{b}$$

- Ask whether the vector can be expressed as a linear combination $\boldsymbol{b} = \boldsymbol{A}\,\boldsymbol{x}$ of another set of vectors $\boldsymbol{A} = (\boldsymbol{a}_1 \ \ \boldsymbol{a}_2 \ \ \dots \boldsymbol{a}_n)$, with coefficients $x_1, x_2, \dots, x_n$.

$$(\ \boldsymbol{A} \,|\, \boldsymbol{b} \ ) = \begin{pmatrix} 1 & 2 & 2 & 15 \\ 2 & 1 & 0 & 4 \\ 1 & 2 & 1 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & 15 \\ 0 & -3 & -4 & -26 \\ 0 & 0 & -1 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & 15 \\ 0 & 3 & 4 & 26 \\ 0 & 0 & 1 & 5 \end{pmatrix} \sim$$

$$\begin{pmatrix} 1 & 2 & 0 & 5 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 15 \\ 4 \\ 10 \end{pmatrix}$$

- A similar procedure can be used for other base changes for example from the basis $\begin{pmatrix} 1 & t & t^2 \end{pmatrix}$ to $\boldsymbol{A} = \begin{pmatrix} 2 & t-1 & t^2-1 \end{pmatrix}$ for $\mathcal{P}_2$. Consider $b(t) = 1 + t + t^2$

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Rightarrow \begin{pmatrix} 2 & t-1 & t^2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & t & t^2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 + t + t^2 \Rightarrow$$

$$\left\{ \begin{array}{lcl} 2x_1 - x_2 - x_3 & = & 1 \\ x_2 & = & 1 \\ x_3 & = & 1 \end{array} \right. \Rightarrow x_1 = \frac{3}{2}$$