Overview

- Matrix rank
- Matrix nullity
- Rank-nullity theorem
- Row-echelon operations to determine rank, nullity

- Consider vectors $\begin{pmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_n \end{pmatrix} = \boldsymbol{A}$ within the Euclidean vector space $(\mathbb{R}^m, +, \mathbb{R}, \cdot)$. Ask: how much of $\mathbb{R}^m$ is within the range of $\boldsymbol{A}$?

Definition. *The (column) rank of a matrix $\boldsymbol{A}$ is the dimension of its range, $\mathcal{R}(\boldsymbol{A})$ (intutively, the number of linearly independent columns), $r = \operatorname{rank}(\boldsymbol{A}) = \dim \mathcal{R}(\boldsymbol{A})$*

Notes:

- $r \leqslant \min(m, n)$, since there are n vectors, and there can't be more linearly independent vectors than m, the dimension of the vector space $\mathbb{R}^m$
- The column rank is the same as the row rank or dimension of $\mathcal{R}(\boldsymbol{A}^T)$[1].

1. The simplest proof is to consider $r > 0$ as the smallest integer for which there exist matrices $\boldsymbol{B} \in \mathbb{R}^{m \times r}$, $\boldsymbol{C} \in \mathbb{R}^{r \times n}$ such that $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C} \Rightarrow \boldsymbol{A}^T = \boldsymbol{C}^T \boldsymbol{B}^T$, and interpret $\boldsymbol{B}$ as the minimal spanning set of $\boldsymbol{A}$, and $\boldsymbol{C}^T$ as the minimal spanning set of $\boldsymbol{A}^T$. Both have r vectors. If no such positive integer exists then $\boldsymbol{A} = \boldsymbol{0}$ of rank 0.

- Recall that the null space of matrix $\boldsymbol{A}$ is $\mathcal{N}(\boldsymbol{A}) = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$.

Definition. *The nullity of a matrix $\boldsymbol{A}$ is the dimension of its null space, $\mathcal{N}(\boldsymbol{A})$ (intutively, the dimension of the space that cannot be reached by linear combination),* $z = \operatorname{null}(\boldsymbol{A}) = \dim \mathcal{N}(\boldsymbol{A})$

The rank-nullity theorem essentially states that a vector can either be reached or not reached by a linear combination, $r + z = n$, there are no other possibilities.

Row-echelon form

- Reduction to row-echelon form can be used to determine matrix rank and nullity. Allowed operations:
  - multiply a row by a scalar
  - interchange rows
  - add a row to another row
- The objective is to form ones on the diagonal. The number of ones is the rank of the matrix.