Overview

- Historical context
- Homogeneous linear second order differential equations:
  - general theory
  - constant coefficients
- Non-homogeneous linear second order differential equations

- The First plague pandemic, known as the Plague of Justinian occurred from 541-542, led to the death of $\sim 20\%$ of the world's population and major political upheaval such as the Anglo-Saxon settlement of Britain, or the decline of the Byzantine Empire in Italy and Arab Peninsula.
- The subsequent flowering of Arab mathematics led to transmission of the Indian number system, mislabeled as "Arab" numerals, and the work of Muhammad ibn Musa al-Khwarizmi, *On the Calculation with Hindu Numerals*, 825
- The Second plague pandemic started with the Black Death of ~ 1350. The plague remained endemic in Eurasia. Its last outbreak in England occurred from 1665 to 1666. Rich Londoners fleed to the countryside. Cambridge advised its students to do the same. Among them a recent Bachelor of Arts graduate: Isaac Newton.
- Newton tried to make sense of the Galileo experiments on falling objects undertaken from 1589 to 1592. The main issue: how to describe a constantly increasing velocity. (https://www.youtube.com/watch?v=RZ1Extu-vcsw, https://www.youtube.com/watch?v=ObPg3ki9GOI)

Theorem. *Assume* p, q *continuus on* (a, b), $x_0 \in (a, b)$, $k_0, k_1 \in \mathbb{R}$. *Then the initial value problem (IVP)*

$$y'' + p(x)\, y' + q(x) y = 0, y(x_0) = k_0, y'(x_0) = k_1$$

has a unique solution on (a, b).

Examples:

1. $y'' - y = 0$
2. $y'' + \omega^2 y = 0$
3. $x^2\, y'' + x\, y' - 4\, y = 0$

Theorem. *If y_1, y_2 are solutions of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

on (a, b) then any linear combination $y = c_1 y_1 + c_2 y_2$ is also a solution

Definition. $\{y_1, y_2\}$ *is a fundamental solution set for (1) if any solution of (1) can be written as a linear combination of y_1, y_2.*

Theorem. *The solutions* y_1, y_2 *of the homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

on (a, b) *form a fundamental set iff* $\{y_1, y_2\}$ *are linearly independent.*

Theorem. $\{y_1, y_2\}$ *are linearly independent if the Wronskian*

$$W = y_1 y_2' - y_1' y_2 = W(x_0) \exp\left[-\int_{x_0}^{x} p(t)\, \mathrm{d}t\right]$$

has no zeros on (a, b)