Overview

- Constant-coefficient homogeneous equations
- Non-homogeneous equations
- Undetermined coefficients

Example 1. $y'' + 6y' + 5y = 0$, $y(0) = 3$, $y'(0) = -1$.

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(%i3) de: 'diff(y,x,2) + 6*'diff(y,x) + 5*y
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(%o3) $\frac{d^2\,y}{d\,x^2} + 6\left(\frac{d\,y}{d\,x}\right) + 5\,y$

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(%i4) gsoln: ode2(de,y,x);
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(%o4) $y = \%k1\,e^{-x} + \%k2\,e^{-5x}$

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(%i6) psoln: ic2(gsoln,x=0,y=3,'diff(y,x)=-1)
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(%o6) $y = \frac{7\,e^{-x}}{2} - \frac{e^{-5x}}{2}$

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(%i11) plot2d(rhs(psoln),[x,0,5],[ylabel,"y(x)"])$
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Example 2. $y'' + 6y' + 9y = 0$, $y(0) = 3$, $y'(0) = -1$.

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(%i12) de: 'diff(y,x,2) + 6*'diff(y,x) + 9*y
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(%o12) $\frac{d^2\, y}{d\, x^2} + 6\left(\frac{d\, y}{d\, x}\right) + 9\, y$

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(%i13) gsoln: ode2(de,y,x);
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(%o13) $y = (\%k2\, x + \%k1)\, \mathrm{e}^{-3x}$

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(%i14) psoln: ic2(gsoln,x=0,y=3,'diff(y,x)=-1)
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(%o14) $y = (8\, x + 3)\, \mathrm{e}^{-3x}$

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(%i15) plot2d(rhs(psoln),[x,0,5],[ylabel,"y(x)"])$
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Example 3. $y'' + 4y' + 13y = 0$, $y(0) = 2$, $y'(0) = -3$.

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(%i16) de: 'diff(y,x,2) + 4*'diff(y,x) + 13*y
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(%o16) $$\frac{d^2\,y}{d\,x^2} + 4\left(\frac{d\,y}{d\,x}\right) + 13\,y$$

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(%i17) gsoln: ode2(de,y,x);
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(%o17) $$y = e^{-2x}\left(\%k1\,\sin\left(3\,x\right) + \%k2\,\cos\left(3\,x\right)\right)$$

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(%i18) psoln: ic2(gsoln,x=0,y=2,'diff(y,x)=-3)
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(%o18) $$y = e^{-2x}\left(\frac{\sin\left(3\,x\right)}{3} + 2\,\cos\left(3\,x\right)\right)$$

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(%i19) plot2d(rhs(psoln),[x,0,5],[ylabel,"y(x)"])$
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Theorem. *Consider* p, q, f *continuous on* (a, b), *and* $x_0 \in (a, b)$, $k_0, k_1 \in \mathbb{R}$. *The initial value problem* $y'' + py' + qy = f$, $y(x_0) = k_0$, $y'(x_0) = k_1$, *has a unique solution on* (a, b).

Theorem. *If* $\{y_1, y_2\}$ *is a fundamental solution set for* $y'' + py' + qy = 0$ *on* (a, b), *and* y_p *is a particular solution of the* $y'' + py' + qy = f$ *on* (a, b), *then the solution of the problem* $y'' + py' + qy = f$, $y(x_0) = k_0$, $y'(x_0) = k_1$, *is of the form*

$$y = y_p + c_1 y_1 + c_2 y_2.$$

Example 4. $y'' + y = 1, y(0), y'(0) = 7$. $y_p = 1$, $\{y_1, y_2\} = \{\cos x, \sin x\}$

$$y = 1 + \cos x + 7 \sin x.$$

Example 5. $y'' - 2y' + y = -3 - x + x^2$, $y(0) = -2$, $y'(0) = 1$.

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(%i24) de: 'diff(y,x,2) - 2*'diff(y,x) + y + 3 + x -x^2
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(%o24) $\frac{d^2 y}{d x^2} - 2\left(\frac{d y}{d x}\right) + y - x^2 + x + 3$

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(%i25) gsoln: ode2(de,y,x);
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(%o25) $y = (\%k1 + \%k2\, x)\, e^x + x^2 + 3\, x + 1$

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(%i26) psoln: ic2(gsoln,x=0,y=-2,'diff(y,x)=1)
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(%o26) $y = (x - 3)\, e^x + x^2 + 3\, x + 1$

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(%i23) plot2d(rhs(psoln),[x,0,5],[ylabel,"y(x)"])$
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Example 6. $x^2\,y'' + x\,y' - 4y = 2x^4$, $y(0) = -2$, $y'(0) = 1$.

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(%i27) de: x^2*'diff(y,x,2) + x*'diff(y,x) -4*y - 2*x^2
```

(%o27) $$x^2\left(\frac{d^2\,y}{d\,x^2}\right) + x\left(\frac{d\,y}{d\,x}\right) - 4\,y - 2\,x^2$$

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(%i28) gsoln: ode2(de,y,x);
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(%o28) $$y = \frac{4\,x^2\log(x) - x^2}{8} + \%k1\,x^2 + \frac{\%k2}{x^2}$$

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(%i30) psoln: ic2(gsoln,x=1,y=1,'diff(y,x)=1)
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(%o30) $$y = \frac{4\,x^2\log(x) - x^2}{8} + \frac{3\,x^2}{4} + \frac{3}{8\,x^2}$$

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(%i32) plot2d(rhs(psoln),[x,1,5],[ylabel,"y(x)"])$
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- Consider equations of form $a\,y'' + b\,y' + c\,y = e^{\alpha x}\,G(x)$, $a, b, c \in \mathbb{R}$. The approach to finding a particular solution is to try the form $y_p = u\,e^{\alpha x}$

1. For $G(x) = g$, a constant, try $y_p = A e^{\alpha x}$

$$A(a\alpha^2 + b\alpha + c) = g, a\alpha^2 + b\alpha + c \neq 0 \Rightarrow A = \frac{g}{a\alpha^2 + b\alpha + c}$$

2. If $a\alpha^2 + b\alpha + c = 0$, try $y_p = A x\, e^{\alpha x}$
3. For general $G(x)$, try $y_p = u(x)\, e^{\alpha x}$

Example 7. $y'' - 3y' + 2y = e^{3x}(x^2 + 2x - 1),\ y(0) = 1,\ y'(0) = 1.$

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(%i33) de:'diff(y,x,2)-3*'diff(y,x)+2*y-exp(3*x)*(x^2+2*x-1)
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(%o33) $$\frac{d^2\,y}{d\,x^2} - 3\left(\frac{d\,y}{d\,x}\right) + 2\,y - \left(-1 + 2\,x + x^2\right) e^{3x}$$

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(%i34) gsoln: ode2(de,y,x);
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(%o34) $$y = \frac{\left(-1 - 2\,x + 2\,x^2\right) e^{3x}}{4} + \%k1\, e^{2x} + \%k2\, e^{x}$$

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(%i36) psoln: ic2(gsoln,x=0,y=1,'diff(y,x)=1)
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(%o36) $$y = \frac{\left(-1 - 2\,x + 2\,x^2\right) e^{3x}}{4} + e^{2x} + \frac{e^{x}}{4}$$

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(%i38) plot2d(rhs(psoln),[x,0,1],[ylabel,"y(x)"])$
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