# Overview

- Spring-damper-mass systems (mechanical displacement storage-loss-inertia)
- $CRL$ system (electric charge storage-loss-inertia)
- Central force systems

- $my'' + ky = 0$
- Characteristic equation $mr^2 + k = 0$, $r_{1,2} = \pm i\omega$, $\omega = \sqrt{k/m}$

$$y(t) = Ae^{i\omega t} + Be^{-i\omega t} = a\cos(\omega t) + b\sin(\omega t)$$

```
(%i1) ode: m*'diff(y,t,2)+k*y
```

(%o1) $m\left(\frac{d^2 y}{d t^2}\right) + k\,y$

```
(%i2) sol: ode2(ode,y,t)
```

Is $k\,m$positive, negative or zero? `positive`

(%o2) $y = \%k1 \sin\left(\frac{\sqrt{k}\,t}{\sqrt{m}}\right) + \%k2 \cos\left(\frac{\sqrt{k}\,t}{\sqrt{m}}\right)$

```
(%i3)
```

# Unforced spring-damper-mass system

- $my'' + cy' + ky = 0$
- Characteristic equation $mr^2 + cr + k = 0$, $r_{1,2} = -\frac{c}{2m} \pm \frac{\sqrt{c^2 - 4km}}{2m}$. Cases:
  - Distinct real roots. $c^2 - 4km > 0$,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

  - Distinct imaginary roots. $c^2 - 4km < 0$,

$$y(t) = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t), \alpha = \frac{c}{2m}, \beta = \frac{\sqrt{4km - c^2}}{2m}.$$

  - Repeated real roots $c^2 - 4km = 0$,

$$y(t) = c_1 e^{rt} + c_2 t e^{rt}, r = -\frac{c}{2m}.$$

**Example.** $y'' - 5y' + 6y = 0$, $y(0) = 1$, $y'(0) = 0$, $r^2 - 5r + 6 = 0 \Rightarrow r_1 = 2, r_3 = 3$

$$y(t) = c_1\, e^{2t} + c_2\, e^{3t},\ y(0) = c_1 + c_2 = 1,\ y'(0) = 2c_1 + 3c_2 = 0.$$

```
(%i1) ode: 'diff(y,t,2)-5*'diff(y,t)+6*y
```

(%o1) $\frac{d^2\, y}{d\, t^2} - 5\left(\frac{d\, y}{d\, t}\right) + 6\, y$

```
(%i2) sol: ode2(ode,y,t)
```

(%o2) $y = \%k1\, \mathrm{e}^{3t} + \%k2\, \mathrm{e}^{2t}$

```
(%i3) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o3) $y = 3\, \mathrm{e}^{2t} - 2\, \mathrm{e}^{3t}$

```
(%i4)
```

**Example.** $y'' + 2y' + 2y = 0,\ y(0) = 1,\ y'(0) = 0,\ r^2 + 2r + 2 = 0 \Rightarrow r_{1,2} = 1 \pm i$

$$\alpha = 1,\ \beta = 1,\ y(t) = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t).$$

```
(%i4) ode: 'diff(y,t,2)+2*'diff(y,t)+2*y
```

(%o4) $\frac{d^2\,y}{d\,t^2} + 2\left(\frac{d\,y}{d\,t}\right) + 2\,y$

```
(%i5) sol: ode2(ode,y,t)
```

(%o5) $y = \mathrm{e}^{-t}\,(\%k1\,\sin(t) + \%k2\,\cos(t))$

```
(%i7) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o7) $y = \mathrm{e}^{-t}\,(\sin(t) + \cos(t))$

```
(%i8)
```

**Example.** $y'' + 2y' + y = 0,\ y(0) = 1,\ y'(0) = 0,\ r^2 + 2r + 1 = 0 \Rightarrow r = -1$

$$y(t) = e^{-rt}(c_1 + c_2 t).$$

```
(%i8) ode: 'diff(y,t,2)+2*'diff(y,t)+y
```

(%o8) $\frac{d^2 y}{d t^2} + 2\left(\frac{d y}{d t}\right) + y$

```
(%i9) sol: ode2(ode,y,t)
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(%o9) $y = (\%k2\, t + \%k1)\, e^{-t}$

```
(%i10) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o10) $y = (t + 1)\, e^{-t}$

```
(%i11)
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- $my''+ky=f(t)$, $mr^2+k=0$, $r_{1,2}=\pm i\omega$, $\omega=\sqrt{k/m}$
- Non-resonant forcing: $f(t)$ not proportional to $\cos(\omega t),\sin(\omega t)$
- Steps:
  1. Solve homogeneous equation $y_h=a\cos(\omega t)+b\sin(\omega t)$
  2. Search for a particular solution suggested by force term

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(%i14) ode: 'diff(y,t,2)+4*y-t
```

(%o14) $\frac{d^2 y}{d t^2}+4\,y-t$

```
(%i15) sol: ode2(ode,y,t)
```

(%o15) $y=\%k1\sin(2\,t)+\%k2\cos(2\,t)+\frac{t}{4}$

```
(%i16) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o16) $$y=-\frac{\sin\left(2\,t\right)}{8}+\cos\left(2\,t\right)+\frac{t}{4}$$

```
(%i17)
```

# Forced mass-spring system, resonant forcing

- $my'' + ky = f(t)$, $mr^2 + k = 0$, $r_{1,2} = \pm i\omega$, $\omega = \sqrt{k/m}$
- Non-resonant forcing: $f(t)$ proportional to $\cos(\omega t), \sin(\omega t)$
- Steps:

  1. Solve homogeneous equation $y_h = a\cos(\omega t) + b\sin(\omega t)$
  2. Search for a particular solution of form $t\,(A\cos\omega t + B\sin\omega t)$

```
(%i17) ode: 'diff(y,t,2)+4*y-sin(2*t)
```

(%o17) $\frac{d^2\,y}{d\,t^2} + 4\,y - \sin(2\,t)$

```
(%i18) sol: ode2(ode,y,t)
```

(%o18) $y = \%k1\sin(2\,t) - \frac{t\cos(2\,t)}{4} + \%k2\cos(2\,t)$

```
(%i13) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o13) $$y = -\frac{17\,e^{3t}}{9} + \frac{11\,e^{2t}}{4} + \frac{5+6\,t}{36}$$

```
(%i14)
```

**Example.** $y'' - 5y' + 6y = \sin t$, $r^2 - 5r + 6 = 0 \Rightarrow r_1 = 2, r_3 = 3$

$$y_h(t) = c_1\, e^{2t} + c_2\, e^{3t},\ y_p = A \sin t + B \cos t.$$

```
(%i19) ode: 'diff(y,t,2)-5*'diff(y,t)+6*y - sin(t)
```

(%o19) $$\frac{d^2\, y}{d\, t^2} - 5 \left(\frac{d\, y}{d\, t}\right) + 6\, y - \sin(t)$$

```
(%i20) sol: ode2(ode,y,t)
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(%o20) $$y = \frac{\cos(t) + \sin(t)}{10} + \%k1\, e^{3t} + \%k2\, e^{2t}$$

```
(%i21) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o21) $$y = \frac{\cos(t) + \sin(t)}{10} - \frac{19\, e^{3t}}{10} + \frac{14\, e^{2t}}{5}$$

```
(%i22)
```

**Example.** $y'' - 5y' + 6y = e^{2t}$, $r^2 - 5r + 6 = 0 \Rightarrow r_1 = 2, r_3 = 3$

$$y_h(t) = c_1\, e^{2t} + c_2\, e^{3t},\ y_p = A e^{2t} + B t e^{2t}.$$

```
(%i22) ode: 'diff(y,t,2)-5*'diff(y,t)+6*y - exp(2*t)
```

(%o22) $\frac{d^2\,y}{d\,t^2} - 5\left(\frac{d\,y}{d\,t}\right) + 6\,y - \mathrm{e}^{2\,t}$

```
(%i23) sol: ode2(ode,y,t)
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(%o23) $y = \%k1\,\mathrm{e}^{3\,t} + (-1-t)\,\mathrm{e}^{2\,t} + \%k2\,\mathrm{e}^{2\,t}$

```
(%i24) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o24) $y = -\mathrm{e}^{3\,t} + (-1-t)\,\mathrm{e}^{2\,t} + 3\,\mathrm{e}^{2\,t}$

```
(%i25)
```

**Example.** $y'' + 2y' + 2y = t$, $r^2 + 2r + 2 = 0 \Rightarrow r_{1,2} = 1 \pm i$

$$\alpha = 1, \beta = 1, y_h(t) = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t), y_p = A + Bt$$

```
(%i25) ode: 'diff(y,t,2)+2*'diff(y,t)+2*y -t
```

(%o25) $$\frac{d^2 y}{d t^2} + 2 \left( \frac{d y}{d t} \right) + 2 y - t$$

```
(%i26) sol: ode2(ode,y,t)
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(%o26) $$y = e^{-t} (\%k2 \cos(t) + \%k1 \sin(t)) + \frac{t-1}{2}$$

```
(%i27) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o27) $$y = e^{-t} \left( \frac{3 \cos(t)}{2} + \sin(t) \right) + \frac{t-1}{2}$$

```
(%i28)
```

**Example.** $y'' + 2y' + 2y = e^{-t}\sin t$, $r^2 + 2r + 2 = 0 \Rightarrow r_{1,2} = 1 \pm i$

$$\alpha = 1,\ \beta = 1,\ y_h(t) = e^{-\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t),\ y_p = t\,e^{-t}(A\cos t + B\sin t)$$

```
(%i31) ode: 'diff(y,t,2)+2*'diff(y,t)+2*y - exp(-t)*sin(t)
```

(%o31) $$\frac{d^2\,y}{d\,t^2} + 2\left(\frac{d\,y}{d\,t}\right) + 2\,y - \mathrm{e}^{-t}\sin(t)$$

```
(%i32) sol: ode2(ode,y,t)
```

(%o32) $$y = \mathrm{e}^{-t}\left(\%k2\cos(t) + \%k1\sin(t)\right) - \frac{t\,\mathrm{e}^{-t}\cos(t)}{2}$$

```
(%i39) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o39) $$y = \mathrm{e}^{-t}\left(\cos(t) + \frac{3\sin(t)}{2}\right) - \frac{t\,\mathrm{e}^{-t}\cos(t)}{2}$$

```
(%i40)
```

**Example.** $y'' + 2y' + y = t$, $r^2 + 2r + 1 = 0 \Rightarrow r = -1$

$$y_h(t) = e^{-rt}(c_1 + c_2 t),\; y_p = A + Bt$$

```
(%i40) ode: 'diff(y,t,2)+2*'diff(y,t)+y - t
```

(%o40) $\frac{d^2 y}{d t^2} + 2\left(\frac{d y}{d t}\right) + y - t$

```
(%i41) sol: ode2(ode,y,t)
```

(%o41) $y = (\%k1 + \%k2\, t)\, e^{-t} + t - 2$

```
(%i42) ic2(sol,t=0,y=1,'diff(y,t)=0)
```

(%o42) $y = (3 + 2\, t)\, e^{-t} + t - 2$

```
(%i43)
```

**Example.** $y''+2y'+y=t\,e^{-t}$, $r^2+2r+1=0 \Rightarrow r=-1$

$$y_h(t)=e^{-rt}(c_1+c_2\,t),\; y_p=(A+Bt)\,t^2\,e^{-rt}$$

```
(%i43) ode: 'diff(y,t,2)+2*'diff(y,t)+y - t*exp(-t)
```

(%o43) $\frac{d^2\,y}{d\,t^2}+2\left(\frac{d\,y}{d\,t}\right)+y-t\,\mathrm{e}^{-t}$

```
(%i44) sol: ode2(ode,y,t)
```

(%o44) $y=\frac{t^3\,\mathrm{e}^{-t}}{6}+(\%k1+\%k2\,t)\,\mathrm{e}^{-t}$

```
(%i45) ic2(sol,t=0,y=1,'diff(y,t)=0)
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(%o45) $y=\frac{t^3\,\mathrm{e}^{-t}}{6}+(1+t)\,\mathrm{e}^{-t}$

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(%i46)
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