Overview

- Motivating examples:
  - Screening clinics
  - SIR, Susceptible, Infectious, Recovered (NLSDE)
  - Coupled oscillators
- Formulation of linear systems of differential equations (LSDE)
- Homogeneous linear systems of differential equations (hLSDE)

Motivating Example 1: Screening clinics

Independent variable	t
Dependent variables	$A(t), B(t)$
	$A, B\colon \mathbb{R} \to \mathbb{R}$

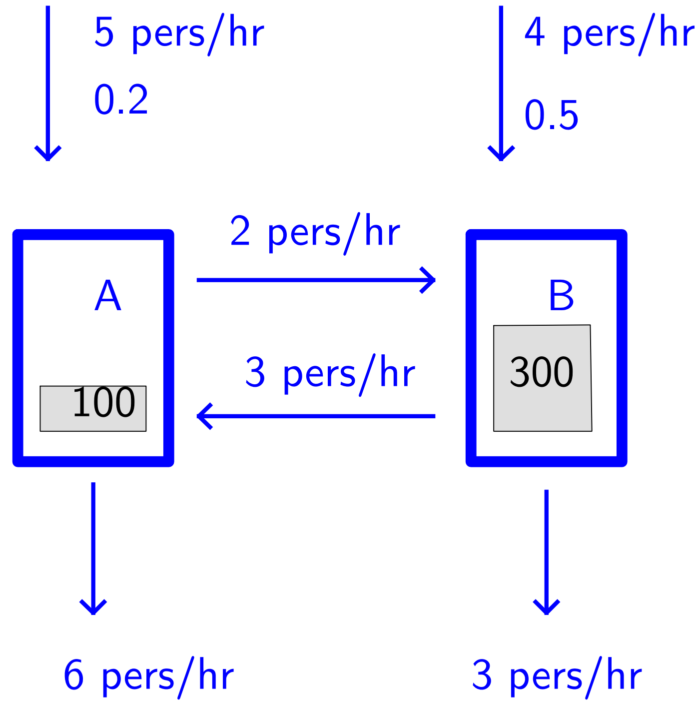
$$A' = 0.2 \times 5 - \frac{8}{100}A + \frac{3}{300}B$$

$$B' = 0.5 \times 4 + \frac{2}{100}A - \frac{6}{300}B$$

$$\begin{cases} A' = 1 - .08A + .01B \\ B' = 2 + .02A - .02B \end{cases}$$

$$\boldsymbol{y}(t) = \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}, \boldsymbol{y}' = \boldsymbol{M}\boldsymbol{y} + \boldsymbol{f}$$

$$\boldsymbol{M} = \begin{pmatrix} -.08 & .01 \\ .02 & -.02 \end{pmatrix} \quad \boldsymbol{f} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$



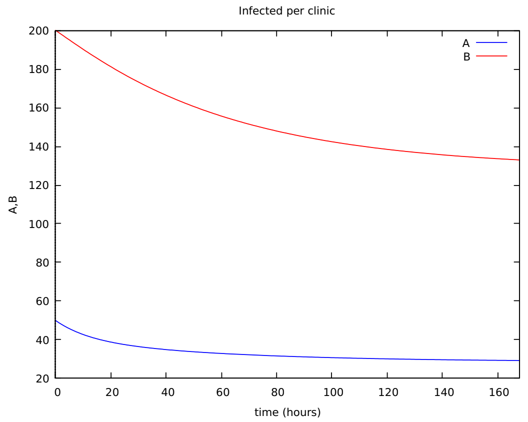
Screening clinics model: solution

```
(%i8) dA: 'diff(A(t),t) = 1 - 8*A(t)/100 + B(t)/100;
      dB: 'diff(B(t),t) = 2 + 2*A(t)/100 - 2*B(t)/100;
```

(%o8) $$\frac{d}{d\,t}\,A(t) = \frac{B(t)}{100} - \frac{2\,A(t)}{25} + 1$$

(%o9) $$\frac{d}{d\,t}\,B(t) = -\frac{B(t)}{50} + \frac{A(t)}{50} + 2$$

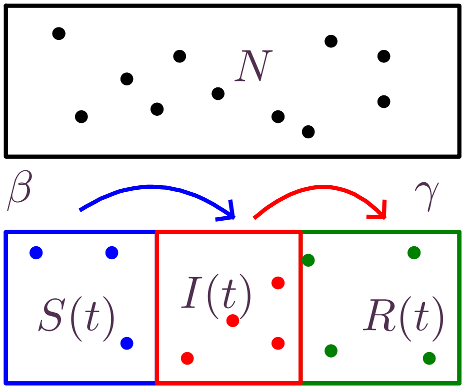
```
(%i30) gsoln: desolve([dA,dB],[A(t),B(t)])$
       psoln: subst([A(0)=50,B(0)=200],gsoln)$
       PA: rhs(psoln[1])$ PB: rhs(psoln[2])$
       plot2d([PA,PB],[t,0,7*24],
              [xlabel,"time (hours)"],[ylabel,"A,B"],
              [title,"Infected per clinic"],
              [legend,"A","B"])$
(%i35)
```



Motivating Example 2: SIR

$$S' = -\beta I S$$
$$I' = \beta I S - \gamma I$$
$$R' = \gamma I$$
$$S, I, R \colon \mathbb{R} \to \mathbb{R}$$

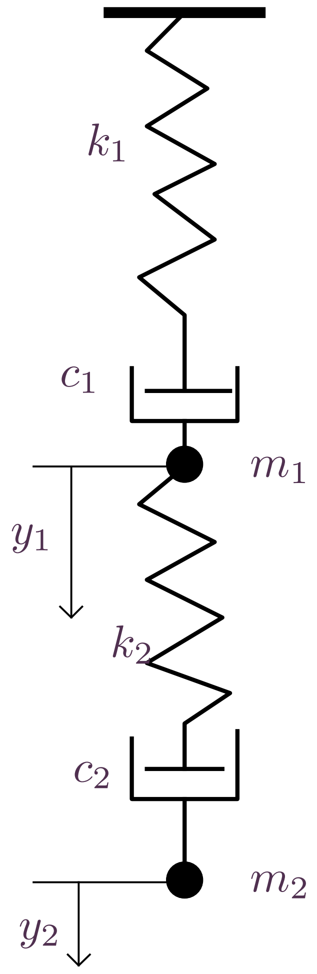
```
(%i1) s:rk([-0.00005*I*S,0.00005*I*S-0.03*I,
      0.03*I],
      [S,I,R],[999,1,0],[t,0,180,1/12])$
      pl: [];
      pl: endcons([discrete,makelist([p[1],
      p[2]],p,s)],pl);
      pl: endcons([discrete,makelist([p[1],
      p[3]],p,s)],pl);
      pl: endcons([discrete,makelist([p[1],
      p[4]],p,s)],pl);
      plot2d(pl,[xlabel,"days"],[ylabel,"S"],
      [title,"Infection propagation"],
      [legend,"S","I","R"])$

(%o26) []

(%i71)
```



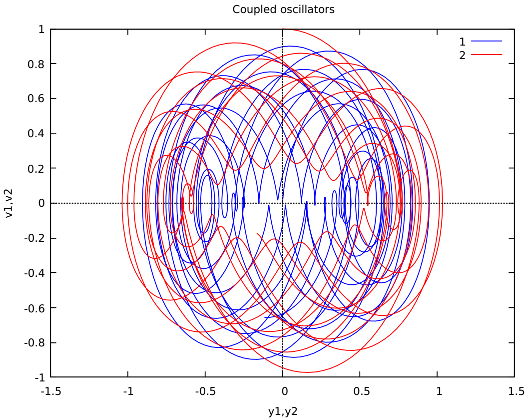
$$m_1\, y_1'' = m_1 g - k_1 y_1 - c_1 y_1' + k_2(y_2 - y_1)$$
$$m_2\, y_2'' = m_2 g - k_2(y_2 - y_1) - c_2 y_2'$$

$$v_1' = g - \kappa_1\, y_1 - \gamma_1 v_1 + \kappa_2(y_2 - y_1)$$
$$v_2' = g - \kappa_2(y_2 - y_1) - \gamma_2\, v_2$$
$$y_1' = v_1$$
$$y_2' = v_2$$

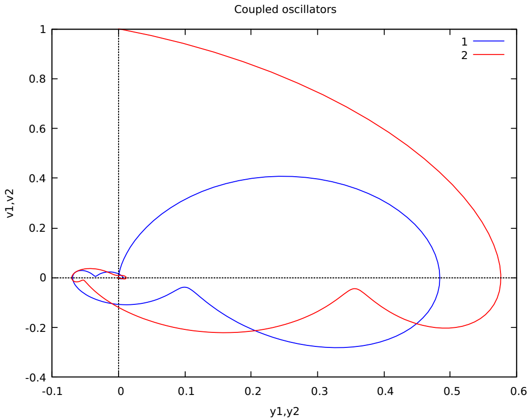
```
(%i121) sol: rk([-y1-v1/100+2*(y2-y1),-2*(y2-y1)-v2/200,v1,v2],
        [v1,v2,y1,y2],[0,1,0,0],[t,0,100,.05])$
        pl: []$
        pl: endcons([discrete,makelist([p[4],p[2]],p,sol)],pl)$
        pl: endcons([discrete,makelist([p[5],p[3]],p,sol)],pl)$
        plot2d(pl,[xlabel,"y1,y2"],[ylabel,"v1,v2"],[title,"Coupled
        oscillators"],
        [legend,"1","2"])$
(%i126)
```



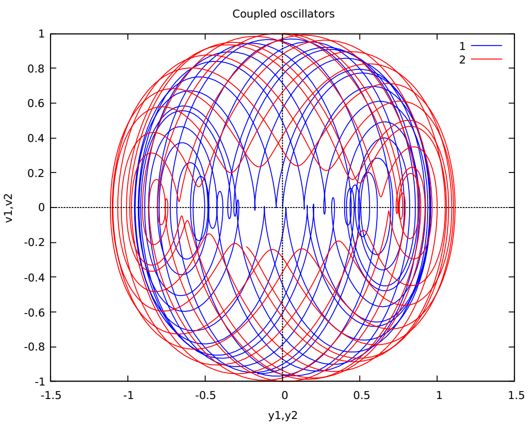
Coupled oscillator solutions



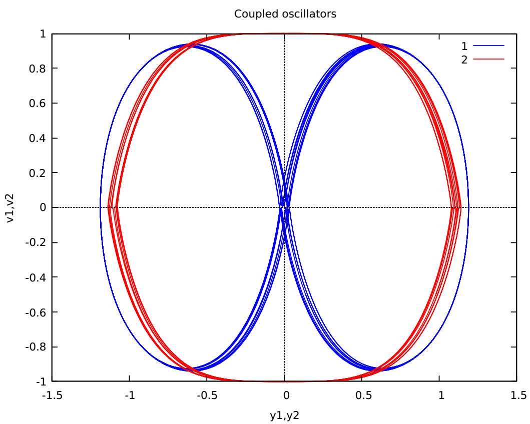
Slightly damped motion



Strongly damped motion



No dampening



Nonlinear springs

- Dependent variables $\boldsymbol{y}\colon \mathbb{R} \to \mathbb{R}^n$

$$(\text{IVP})\ \boldsymbol{y}(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \boldsymbol{y}' = \boldsymbol{A}(t)\, \boldsymbol{y} + \boldsymbol{f}(t) = \boldsymbol{F}(t, \boldsymbol{y}), \boldsymbol{y}(0) = \boldsymbol{k} = \boldsymbol{y}_0.$$

Theorem. *If $\boldsymbol{A}\colon \mathbb{R} \to \mathbb{R}^{n\times n}$ and $\boldsymbol{f}\colon \mathbb{R} \to \mathbb{R}^n$ are continuous on (a, b) that contains the initial condition, then the initial value problem (IVP) has a unique solution on (a, b).*

- $z^{(n)} + p_1\, z^{(n-1)} + \cdots + p_{n-1} z' + p_n z = -g(t)$

$$z = y_1, z' = y_2 = y_1', \ldots, z^{(n-1)} = y_n = y_{n-1}'$$

$$y_n' = -g - p_1 y_n - \ldots - p_n y_1.$$

- $\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y}$ is a homogeneous LSDE (hLSDE). $\boldsymbol{y} = \boldsymbol{0}$ is a trivial solution.
- $\mathcal{S} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n\}$. Linear combination

$$(\mathrm{LC})\ \boldsymbol{y} = c_1\,\boldsymbol{y}_1 + c_2\,\boldsymbol{y}_2 + \ldots + c_n\,\boldsymbol{y}_n.$$

- If any solution of $\boldsymbol{y}' = \boldsymbol{A}\,\boldsymbol{y}$ can be written as the linear combination (LC), then $\mathcal{S}$ is a *fundamental set of solutions*.

Theorem. *The set of solutions $\mathcal{S} = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n\}$ is a fundamental set if and only if $\mathcal{S}$ is linearly independent.*