Overview

- Fundamental set of solutions (Wronskian)
- Constant-coefficient hLSDEs, matrix eigenvalue problem
- Cases:
  - Real negative eigenvalues
  - Real positive eigenvalues
  - Purely imaginary eigenvalues
  - Complex eigenvalues

- $\boldsymbol{y}' = \boldsymbol{A}(t)\,\boldsymbol{y}$, $\boldsymbol{y}(t)$, $\boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n$
- Set of solutions $\mathcal{S} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$, $\boldsymbol{y}_i' = \boldsymbol{A}\boldsymbol{y}_i$, $i = 1, \ldots, n$
- $\mathcal{S}$ is a *fundamental set of solutions* of $\{\boldsymbol{y}_1(t), \ldots, \boldsymbol{y}_n(t)\}$ are linearly independent (for all t)
- $\boldsymbol{y} = c_1\boldsymbol{y}_1 + \cdots + c_n\boldsymbol{y}_n$, $c_1\boldsymbol{y}_1 + \cdots + c_n\boldsymbol{y}_n = \boldsymbol{0} \Rightarrow c_1 = \cdots = c_n = 0$.

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 & \ldots & \boldsymbol{y}_n \end{pmatrix}\boldsymbol{c} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \boldsymbol{Y}\boldsymbol{c} = \boldsymbol{y}$$

- $\boldsymbol{Y}\boldsymbol{c} = \boldsymbol{y}$ always has a solution iff $\mathcal{R}(\boldsymbol{Y}) = \mathbb{R}^n$, or equivalently if $\det(\boldsymbol{Y}) \neq 0$
- The *Wronskian* of $\mathcal{S} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$ is $W(t) = \det(\boldsymbol{Y}(t))$,$W: \mathbb{R} \to \mathbb{R}$.

Theorem. *Abel's Formula. If $\boldsymbol{A}(t)$ is continuous on (a, b), and $\mathcal{S} = \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n\}$ is a set of solutions of $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ on (a, b), then the Wronskian of $\mathcal{S}$ is given by*

$$W(t) = \exp\left[\int_{t_0}^{t} \operatorname{tr} \boldsymbol{A}(s)\,\mathrm{d}s\right] W(t_0). \boldsymbol{Y}' = \begin{pmatrix} \boldsymbol{y}_1' & \ldots & \boldsymbol{y}_n' \end{pmatrix} = \boldsymbol{A}\boldsymbol{Y}$$

- $y\colon \mathbb{R}\to\mathbb{R}$, $y' = a(t)\, y \Rightarrow y(t) = \exp\left[\int_{t_0}^{t} a(s)\,\mathrm{d}s\right] y(t_0)$

$$y'(t) = \left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right)' \exp\left[\int_{t_0}^{t} a(s)\,\mathrm{d}s\right] y(t_0) = a(t)\, y$$

$$\exp(a) = \frac{1}{0!}a^0 + \frac{1}{1!}a^1 + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots = 1 + a + \frac{1}{2}a^2 + \frac{1}{6}a^3 + \cdots$$

- $\boldsymbol{y}\colon \mathbb{R}\to\mathbb{R}^n$, $\boldsymbol{y}' = \boldsymbol{A}(t)\,\boldsymbol{y}$, $\boldsymbol{y}(t) = \exp\left[\int_{t_0}^{t} \boldsymbol{A}(s)\,\mathrm{d}s\right]\boldsymbol{y}(t_0)$,

$$\boldsymbol{y}'(t) = \left(\int_{t_0}^{t} \boldsymbol{A}(s)\,\mathrm{d}s\right)' \exp\left[\int_{t_0}^{t} \boldsymbol{A}(s)\,\mathrm{d}s\right] \boldsymbol{y}(t_0) = \boldsymbol{A}(t)\,\boldsymbol{y}$$

$$\exp(\boldsymbol{A}) = \frac{1}{0!}\boldsymbol{A}^0 + \frac{1}{1!}\boldsymbol{A}^1 + \frac{1}{2!}\boldsymbol{A}^2 + \frac{1}{3!}\boldsymbol{A}^3 + \cdots = \boldsymbol{I} + \boldsymbol{A} + \frac{1}{2}\boldsymbol{A}^2 + \frac{1}{6}\boldsymbol{A}^3 + \cdots$$

- trace of a matrix

$$\operatorname{tr}\boldsymbol{A}(t) = a_{11}(t) + \cdots + a_{nn}(t) = \sum_{i=1}^{n} a_{ii}(t)$$

Example 1. Check is $\mathcal{S} = \{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ is a fundamental set of solutions for $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$

$$\boldsymbol{y}_1 = \begin{pmatrix} -e^{2t} \\ 2e^{2t} \end{pmatrix}, \boldsymbol{y}_2 = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} -4 & -3 \\ 6 & 5 \end{pmatrix}$$

Solution 1. Solve $c_1\,\boldsymbol{y}_1 + c_2\,\boldsymbol{y}_2 = \boldsymbol{0}$

$$c_1\,\boldsymbol{y}_1 + c_2\,\boldsymbol{y}_2 = c_1\begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2\begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} = \begin{pmatrix} -c_1e^{2t} - c_2e^{-t} \\ 2c_1e^{2t} + c_2e^{-t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_1 = c_2 = 0$$

Solution 2. Directly compute the Wronskian

$$W(t) = \det(\boldsymbol{Y}) = |\ \boldsymbol{y}_1 \quad \boldsymbol{y}_2\ | = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^t > 0$$

Solution 3. Apply Abel's formula

$$W(0) = \begin{vmatrix} -1 & -1 \\ 2 & 1 \end{vmatrix} = 1, W(t) = \exp\left[\int_0^t \operatorname{tr} \boldsymbol{A}(s)\,\mathrm{d}s\right] = e^t > 0$$

- Consider $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$. How to find a solution?
- Recall that for $y' = ay$, we'd guess $y(t) = ce^{\lambda t} \Rightarrow \lambda = a$, $y(t) = ce^{\lambda t}$
- Try the same approach for systems, guess that the solution is of the form

$$\boldsymbol{y} = e^{\lambda t}\boldsymbol{x} \Rightarrow \lambda\, e^{\lambda t}\boldsymbol{x} = \boldsymbol{A}(e^{\lambda t}\boldsymbol{x}) \Rightarrow \boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$$

- For $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ finding $\lambda, \boldsymbol{x}$ that satisfy $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ is known as the *eigenvalue problem*
  - λ are the eigenvalues of $\boldsymbol{A}$
  - $\boldsymbol{x}$ are the eigenvectors of $\boldsymbol{A}$ ($\boldsymbol{x} \neq \boldsymbol{0}$, no trivial solutions)
- The eigenvalue relationship can be rewritten as

$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \Rightarrow \boldsymbol{A}\boldsymbol{x} - \lambda\boldsymbol{x} = \boldsymbol{0}, (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0} \Rightarrow p_{\boldsymbol{A}}(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = 0$$

  where $p_{\boldsymbol{A}}(\lambda)$ is the *characteristic polynomial* of $\boldsymbol{A}$, and is of degree n, with n roots.
- Denote the n eigenvalues as $\lambda_1, \ldots, \lambda_n$, and the n eigenvectors as $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$

Characteristic polynomial

- $p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$

$$p_{\boldsymbol{A}}(\lambda) = \left| \begin{pmatrix} \lambda - a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & \lambda - a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & \lambda - a_{nn} \end{pmatrix} \right|$$

- $p_{\boldsymbol{A}}(\lambda) = \lambda^n - \operatorname{tr} \boldsymbol{A} \lambda^{n-1} + \dots + (-1)^n \det \boldsymbol{A} = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0$
- $p_{\boldsymbol{A}}(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_{n-1})(\lambda - \lambda_n) = \lambda^n + (-\lambda_1 - \lambda_2 - \dots - \lambda_n)\lambda^{n-1}$
- Vieta relations

$$\begin{aligned} &\lambda_1 + \lambda_2 + \dots + \lambda_n = -a_{n-1} = \operatorname{tr} \boldsymbol{A} \\ &\lambda_1\lambda_2 + \lambda_1\lambda_3 + \dots + \lambda_{n-1}\lambda_n = a_{n-2} \\ &\dots \\ &\lambda_1\lambda_2\dots\lambda_n = (-1)^n a_0 = (-1)^n \det \boldsymbol{A} \end{aligned}$$

- Solve the eigenvalue problem $\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$
- Form the set of solutions $\mathcal{S} = \{e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n\}$
- Determine is the set of solutions is fundamental:
  - Does $c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \ldots + c_n e^{\lambda_n t}\boldsymbol{x}_n = \boldsymbol{0}$ imply $c_1 = \cdots = c_n = 0$? If so, $\mathcal{S}$ is a fundamental set
  - Compute the Wronskian

$$W(t) = \begin{vmatrix} e^{\lambda_1 t}x_{11} & e^{\lambda_2 t}x_{12} & \ldots & e^{\lambda_n t}x_{1n} \\ e^{\lambda_1 t}x_{21} & e^{\lambda_2 t}x_{22} & \ldots & e^{\lambda_n t}x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\lambda_1 t}x_{n1} & e^{\lambda_2 t}x_{n2} & \ldots & e^{\lambda_n t}x_{\text{nn}} \end{vmatrix} = e^{\lambda_1 t} \ldots e^{\lambda_n t} \begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{\text{nn}} \end{vmatrix}$$

  and if $W(t) \neq 0$ for any t, $\mathcal{S}$ is a fundamental set of solutions

- Eigenvector matrix

$$\boldsymbol{X} = (\ \boldsymbol{x}_1 \ \ \boldsymbol{x}_2 \ \ \ldots \ \ \boldsymbol{x}_n\ ) = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{\text{nn}} \end{pmatrix}$$

- Suppose $\lambda_1, \ldots, \lambda_n < 0$, and $\mathcal{S} = \{e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n\}$ is a fundamental set

$$\boldsymbol{y} = c_1\, e^{\lambda_1 t}\,\boldsymbol{x}_1 + \ldots + c_n\, e^{\lambda_n t}\boldsymbol{x}_n$$

- Then $\lim_{t\to\infty}\boldsymbol{y}(t) = \boldsymbol{0}$

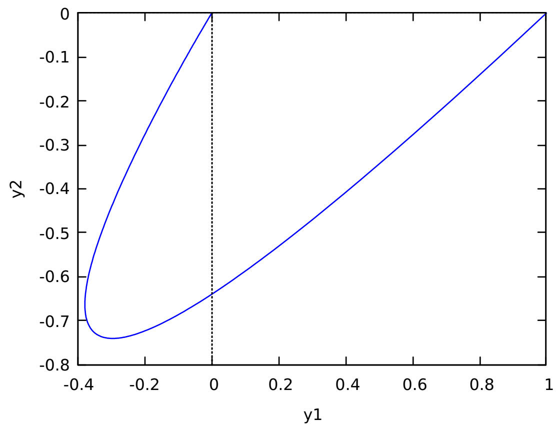
```
(%i14) A: matrix([-7,4],[-5,2])$
       p: factor(charpoly(A,lambda));
```

(%o15) $(\lambda+2)\,(\lambda+3)$

```
(%i3) y: matrix([y1(t)],[y2(t)])$ rhs: A.y$
      dy: matrix(['diff(y1(t),t)],['diff(y2(t),t)])$
      eq1: dy[1][1] = rhs[1][1]$
      eq2: dy[2][1] = rhs[2][1]$
      sys: [eq1,eq2]$
      gsoln: desolve(sys,[y1(t),y2(t)])$
      psoln: subst([y1(0)=1,y2(0)=0],gsoln);
```

(%o10) $[y1(t) = 5\,\mathrm{e}^{-3t} - 4\,\mathrm{e}^{-2t},\, y2(t) = 5\,\mathrm{e}^{-3t} - 5\,\mathrm{e}^{-2t}]$

```
(%i11) plot2d([parametric,rhs(psoln[1]),rhs(psoln[2]),
              [t,0,10]],[xlabel,"y1"],[ylabel,"y2"])$
(%i12)
```



- Suppose one eigenvalue is real positive, and $\mathcal{S} = \{e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n\}$ is a fundamental set

$$\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + \ldots + c_n e^{\lambda_n t}\boldsymbol{x}_n$$

- Then generally $\boldsymbol{y}(t) \to \infty$ (see Lesson 20 for discussion)

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(%i111) A: matrix([2,4],[4,2])$
        p: factor(charpoly(A,lambda));
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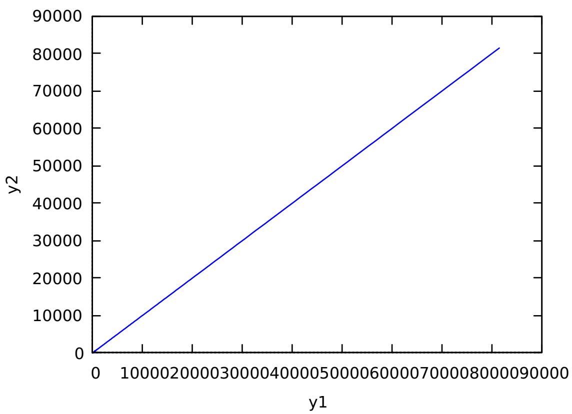
(%o112) $(\lambda - 6)(\lambda + 2)$

```
(%i113) y: matrix([y1(t)],[y2(t)])$ rhs: A.y$
        dy: matrix(['diff(y1(t),t)],['diff(y2(t),t)])$
        eq1: dy[1][1] = rhs[1][1]$
        eq2: dy[2][1] = rhs[2][1]$
        sys: [eq1,eq2]$
        gsoln: desolve(sys,[y1(t),y2(t)])$
        psoln: subst([y1(0)=1,y2(0)=0],gsoln);
```

(%o120) $\left[y1(t) = \frac{e^{6t}}{2} + \frac{e^{-2t}}{2}, y2(t) = \frac{e^{6t}}{2} - \frac{e^{-2t}}{2}\right]$

```
(%i122) plot2d([parametric,rhs(psoln[1]),rhs(psoln[2]),
                [t,0,2]],[xlabel,"y1"],[ylabel,"y2"])$
```



- Suppose all eigenvalues are purely imaginary, $\mathcal{S} = \{e^{\lambda_1 t}\boldsymbol{x}_1, \ldots, e^{\lambda_n t}\boldsymbol{x}_n\}$ is a fundamental set

$$\boldsymbol{y} = c_1\, e^{\lambda_1 t}\, \boldsymbol{x}_1 + \ldots + c_n\, e^{\lambda_n t} \boldsymbol{x}_n$$

- Then $\boldsymbol{y}(t)$ exhibits cycles

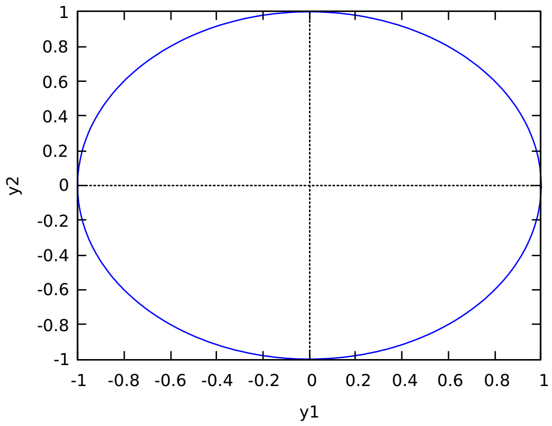
```
(%i12) A: matrix([0,1],[-1,0])$
       p: factor(charpoly(A,lambda));
```

(%o13) $\lambda^2 + 1$

```
(%i14) y: matrix([y1(t)],[y2(t)])$ rhs: A.y$
       dy: matrix(['diff(y1(t),t)],['diff(y2(t),t)])$
       eq1: dy[1][1] = rhs[1][1]$
       eq2: dy[2][1] = rhs[2][1]$
       sys: [eq1,eq2]$
       gsoln: desolve(sys,[y1(t),y2(t)])$
       psoln: subst([y1(0)=1,y2(0)=0],gsoln);
```

(%o21) $[y1(t) = \cos(t), y2(t) = -\sin(t)]$

```
(%i22) plot2d([parametric,rhs(psoln[1]),rhs(psoln[2]),
              [t,0,6.3]],[xlabel,"y1"],[ylabel,"y2"])$
(%i23)
```



- Suppose all eigenvalues are complex, $\mathcal{S}=\{e^{\lambda_1 t}\boldsymbol{x}_1,\ldots,e^{\lambda_n t}\boldsymbol{x}_n\}$ is a fundamental set

$$\boldsymbol{y}=c_1\,e^{\lambda_1 t}\,\boldsymbol{x}_1+\ldots+c_n\,e^{\lambda_n t}\boldsymbol{x}_n$$

- Then $\boldsymbol{y}(t)$ exhibits spirals, inward towards zero for negative real part, outwards towards infinity for positive real part

```
(%i60) A: matrix([-1,-10],[10,-1])$
       p: factor(charpoly(A,lambda));
```

(%o61) $\lambda^2+2\,\lambda+101$

```
(%i62) y: matrix([y1(t)],[y2(t)])$ rhs: A.y$
       dy: matrix(['diff(y1(t),t)],['diff(y2(t),t)])$
       eq1: dy[1][1] = rhs[1][1]$
       eq2: dy[2][1] = rhs[2][1]$
       sys: [eq1,eq2]$
       gsoln: desolve(sys,[y1(t),y2(t)])$
       psoln: subst([y1(0)=1,y2(0)=0],gsoln);
```

(%o69) $[y1(t)=\mathrm{e}^{-t}\cos(10\,t),\,y2(t)=\mathrm{e}^{-t}\sin(10\,t)]$

```
(%i70) plot2d([parametric,rhs(psoln[1]),rhs(psoln[2]),
              [t,0,24]],[xlabel,"y1"],[ylabel,"y2"])$
(%i71)
```