Overview

- Direction fields
- Phase portraits
- Repeated eigenvalues

- In $\mathbb{R}^n$ the hLSDE $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ defines a *direction field* $\boldsymbol{y}' = \left(y_1' \; \ldots \; y_n' \right)$, useful for qualitative characterization of the solution from knowledge of eigenvalues of $\boldsymbol{A}$
- Mass-dampener-spring system $mu'' + cu' + ku = 0$, set $v = u'$, $m > 0$, $2\gamma = c/m$, $\kappa = k/m$

$$\begin{cases} u' = v \\ v' = -\kappa u - 2\gamma v \end{cases}, \boldsymbol{y} = \begin{pmatrix} u \\ v \end{pmatrix}, \boldsymbol{y}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}, \boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\kappa & -2\gamma \end{pmatrix}$$

- Characteristic polynomial

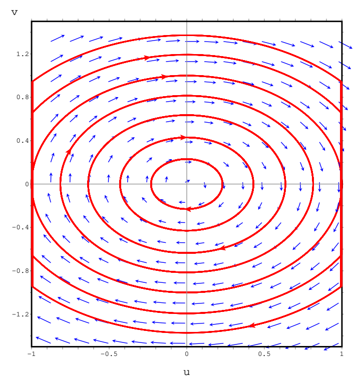
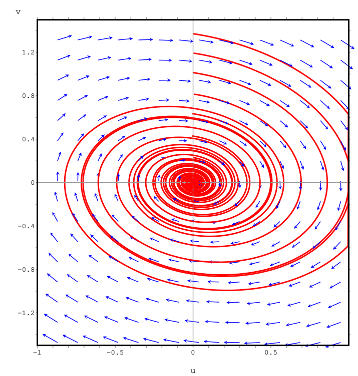
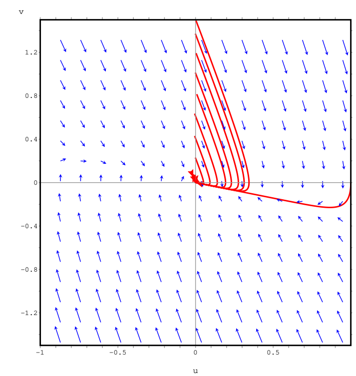
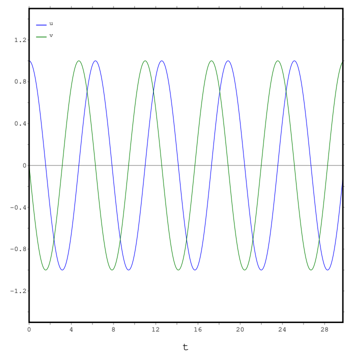
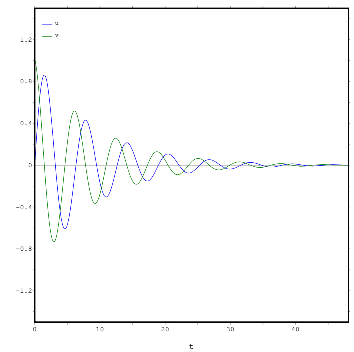
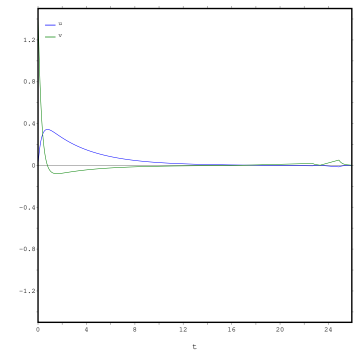
$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ \kappa & \lambda + 2\gamma \end{vmatrix} = \lambda^2 + 2\gamma\lambda + \kappa$$

- Roots of characteristic polynomial $\lambda_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \kappa^2}$, $\gamma \geqslant 0$, $\kappa > 0$, $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$

$$\text{No dampening: } \gamma = 0, \quad \text{Under} - \text{damped: } \kappa > \gamma > 0 \quad \text{Over} - \text{damped: } \gamma > \kappa$$

```
(%i9) plotdf([v,-(kappa*u+2*gamma*v)],[u,v],[trajectory_at,1,0],[u,-1,1],[v,-1.5,1.5],
      [nsteps,2500],[tstep,.001],[direction,forward],[parameters,"kappa=1,gamma=0"],
      [sliders,"kappa=.1:2,gamma=0:4"],[versus_t,1])$
(%i10)
```

Mass-dampener-spring system

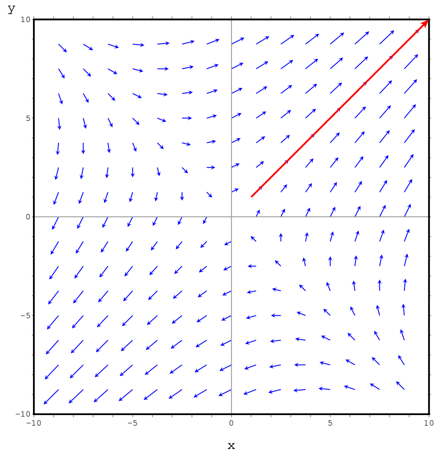
	No dampening	Under-damped	Over-damped
$(u(t), v(t))$			
$\{(t, u(t)), (t, v(t))\}$			

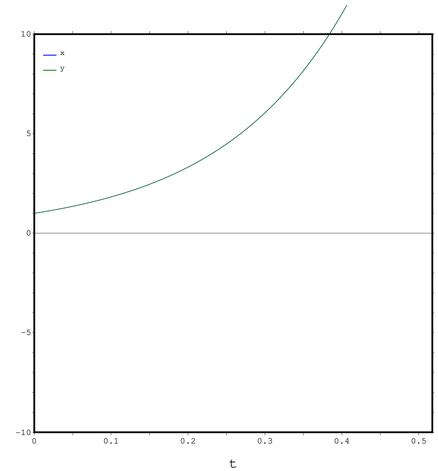
- All eigenvalues negative $\Rightarrow$ phase trajectories go to zero, $\boldsymbol{y} = c_1\, e^{\lambda_1 t} \boldsymbol{x}_1 + \ldots + c_n\, e^{\lambda_n t} \boldsymbol{x}_n$

```
(%i18) A: matrix([-7,4],[-5,2])$ Y: matrix([x],[y])$
       Yp: A.Y$ p: factor(charpoly(A,lambda));
```

(%o21) $(\lambda+2)\,(\lambda+3)$

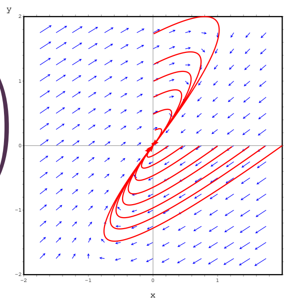
```
(%i17) plotdf([Yp[1][1],Yp[2][1]],[trajectory_at,2,0],
       [x,-2,2],[y,-2,2],[direction,forward],
       [versus_t,1])$
(%i18)
```





$$\boldsymbol{y} = c_1\, e^{-2t} \boldsymbol{x}_1 + c_2\, e^{-3t} \boldsymbol{x}_2$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$



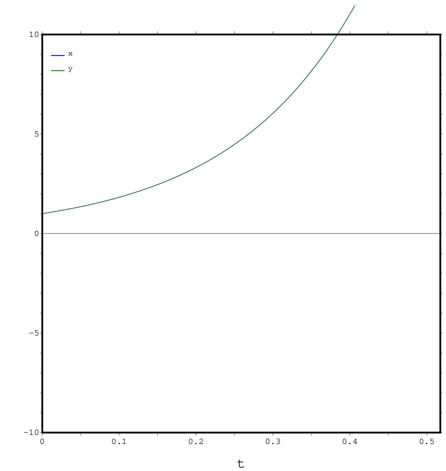
$$\frac{y_1}{y_2} = \frac{c_1\, x_{11}\, e^{-2t} + c_2\, x_{12}\, e^{-3t}}{c_1\, x_{21}\, e^{-2t} + c_2\, x_{22}\, e^{-3t}}$$

$$\frac{y_1}{y_2} = \frac{x_{11} + (c_2 / c_1)\, x_{12}\, e^{-t}}{x_{21} + (c_2 / c_1)\, x_{22}\, e^{-t}}$$

$$\lim_{t \to \infty} \frac{y_1}{y_2} = \frac{x_{11}}{x_{21}}$$

One positive real eigenvalue

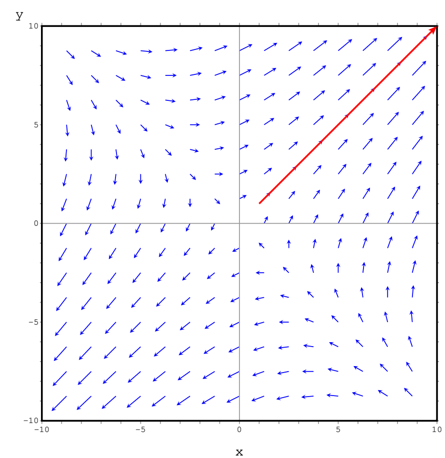
- One positive eigenvalues (say, $\lambda_1 > 0$) $\Rightarrow$ phase trajectories go to ∞ if $c_1 \neq 0$, $\boldsymbol{y} = c_1\, e^{\lambda_1 t} \boldsymbol{x}_1 + c_2\, e^{\lambda_2 t} \boldsymbol{x}_2$




```
(%i44) A: matrix([2,4],[4,2])$ Y: matrix([x],[y])$ Yp:
       A.Y$ p: factor(charpoly(A,lambda));
```

(%o47) $(\lambda - 6)\,(\lambda + 2)$

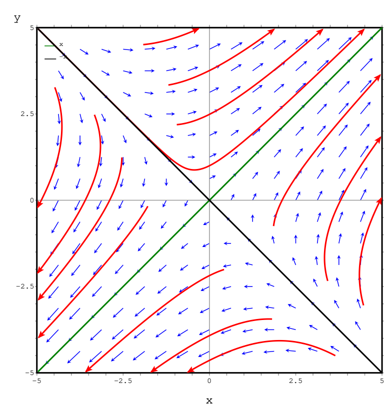
```
(%i39) plotdf([Yp[1][1],Yp[2][1]],
       [xfun,"x;-x"],
       [trajectory_at,-5,5.001],
       [x,-5,5],[y,-5,5],[direction,forward],
       [versus_t,1])$
(%i40)
```



$$\boldsymbol{y} = c_1\, e^{6t} \boldsymbol{x}_1 + c_2\, e^{-2t} \boldsymbol{x}_2$$

$$c_1 \neq 0 \Rightarrow \lim_{t \to \infty} \boldsymbol{y} = \infty$$

$$c_1 = 0 \Rightarrow \lim_{t \to \infty} \boldsymbol{y} = 0$$

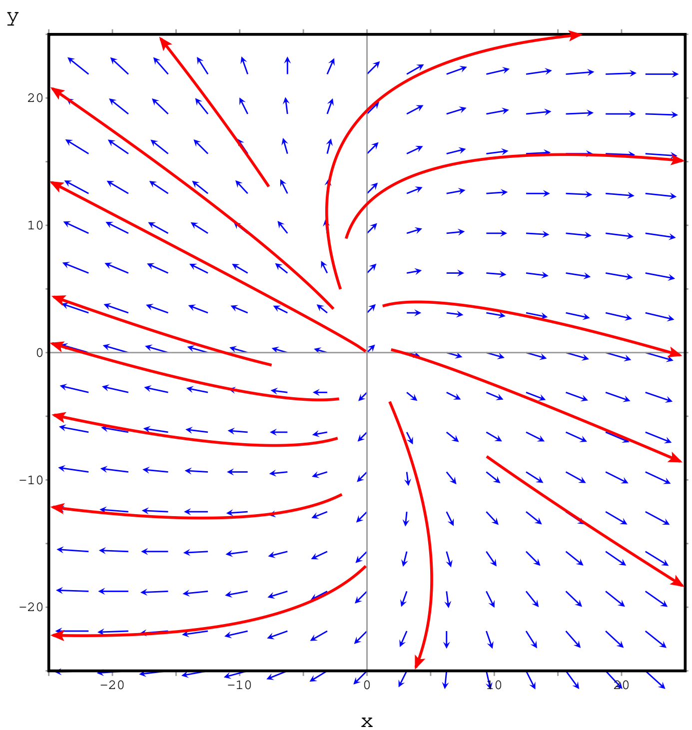


- Two positive eigenvalues $\Rightarrow$ phase trajectories go to ∞, $\boldsymbol{y} = c_1\, e^{6t}\, \boldsymbol{x}_1 + c_2\, e^{3t} \boldsymbol{x}_2$

```
(%i150) A: matrix([7,2],[-2,2])$
        Y: matrix([x],[y])$ Yp: A.Y$ p:
        factor(charpoly(A,lambda));
```

(%o153) $(\lambda - 6)\,(\lambda - 3)$

```
(%i154) plotdf([Yp[1][1],Yp[2][1]],
         [trajectory_at,-.1,.1],
         [x,-25,25],[y,-25,25],
         [direction,forward],
         [versus_t,1])$
(%i155)
```



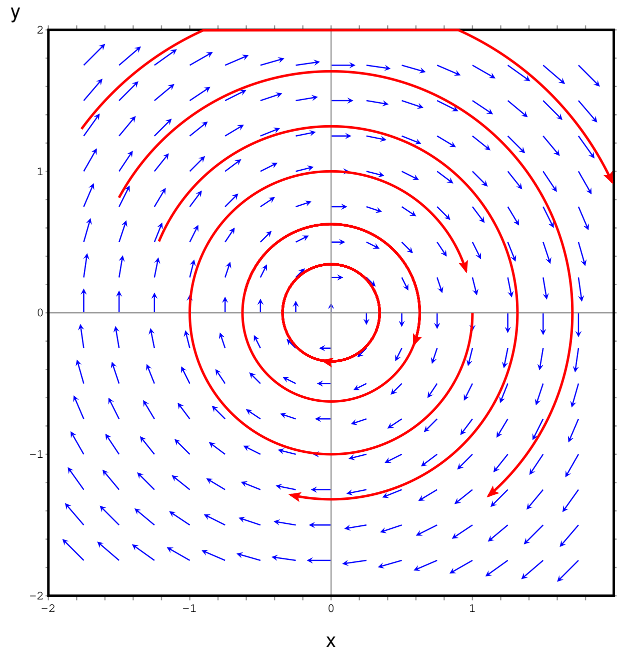
- Imaginary eigenvalues $\Rightarrow$ cycle $\boldsymbol{y} = c_1\, e^{2\mathrm{i}t}\, \boldsymbol{x}_1 + c_2\, e^{-2\mathrm{i}t} \boldsymbol{x}_2 = a_1 \cos(2t)\, \boldsymbol{x}_1 + a_2 \sin(2t)\, \boldsymbol{x}_2$

```
(%i155) A: matrix([0,2],[-2,0])$
        Y: matrix([x],[y])$ Yp: A.Y$ p:
        factor(charpoly(A,lambda));
```

(%o158) $\lambda^2 + 4$

```
(%i159)
```

```
(%i159) plotdf([Yp[1][1],Yp[2][1]],
        [trajectory_at,1,0],
        [x,-2,2],[y,-2,2],[direction,
        forward],
        [versus_t,1])$
(%i160)
```



- Complex conjugate eigenvalues $\Rightarrow$ $\boldsymbol{y} = e^{\alpha t}[c_1 \cos(\beta t)\, \boldsymbol{x}_1 + c_2 \sin(\beta t)\, \boldsymbol{x}_2]$

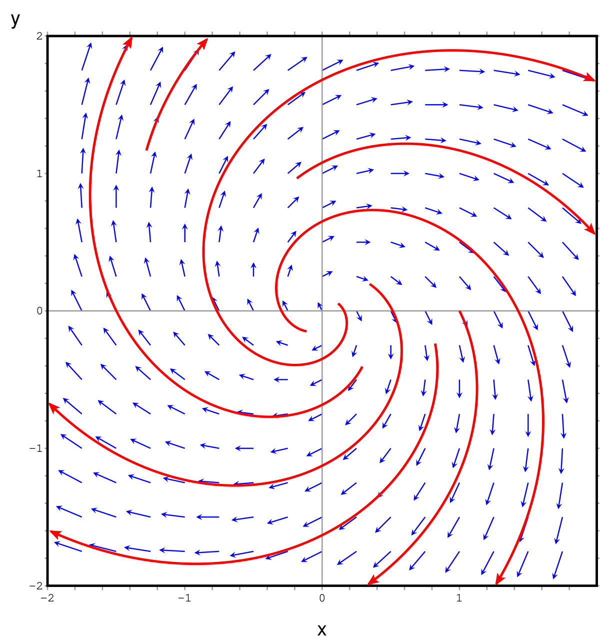
```
(%i160) A: matrix([1,2],[-2,1])$
        Y: matrix([x],[y])$ Yp: A.Y$ p:
        factor(charpoly(A,lambda));
```

(%o163) $\lambda^2 - 2\,\lambda + 5$

```
(%i164)
```

```
(%i164) plotdf([Yp[1][1],Yp[2][1]],
        [trajectory_at,1,0],
        [x,-2,2],[y,-2,2],[direction,
        forward],
        [versus_t,1])$
```

```
(%i165)
```



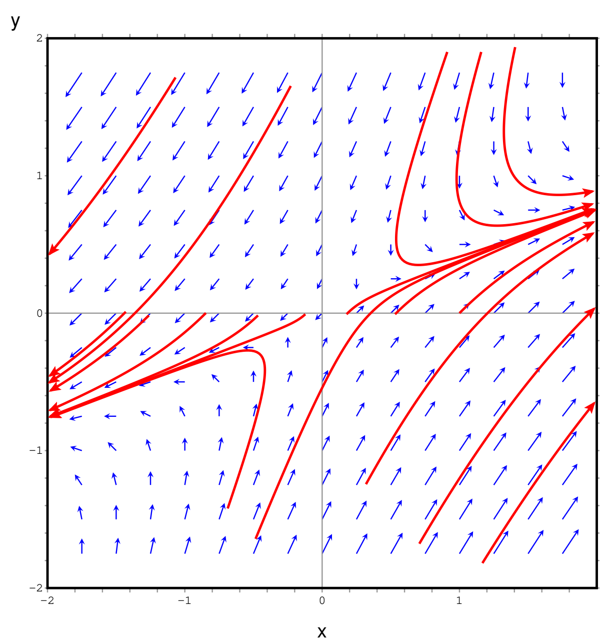
- All real eigenvalues

```
(%i165) A:matrix([1,-1,-2],[1,-2,-3],[-4,1,-1])$
        Y:matrix([x],[y],[z])$ Yp: A.Y$ p:
        factor(charpoly(A,lambda));
```

(%o168) $-(\lambda - 2)(\lambda + 1)(\lambda + 3)$

```
(%i169)
```

```
(%i183) plotdf([Yp[1][1],Yp[2][1]],
        [trajectory_at,1,0],
        [parameters,"z=0"],
        [sliders,"z=-2:2"],
        [x,-2,2],[y,-2,2],[direction,
        forward])$
(%i184)
```



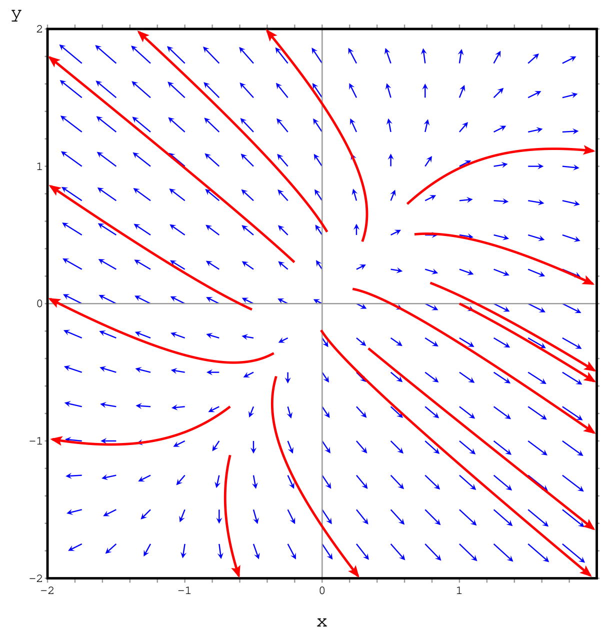
- one complex conjugate pair

```
(%i177) A:matrix([4,-2,-2],[-2,3,1],[2,-1,3])$
        Y:matrix([x],[y],[z])$ Yp: A.Y$ p:
        factor(charpoly(A,lambda));
```

(%o180) $-(\lambda^3 - 10\,\lambda^2 + 34\,\lambda - 32)$

```
(%i181)
```

```
(%i184) plotdf([Yp[1][1],Yp[2][1]],
        [trajectory_at,1,0],
        [parameters,"z=0"],
        [sliders,"z=-2:2"],
        [x,-2,2],[y,-2,2],[direction,
        forward])$
(%i185)
```



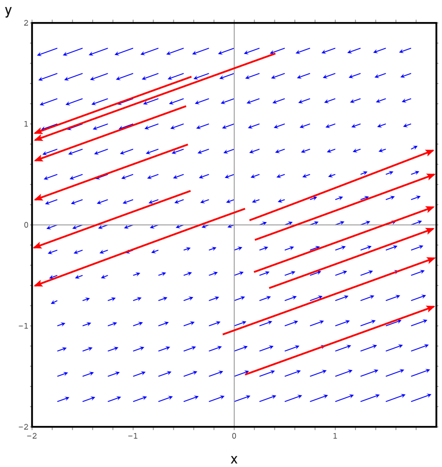
- If eigenvalue λ_k is an $m-$ multiple root of the characteristic polynomial, the independent solutions become

$$y_k = e^{\lambda_k t},\, y_{k+1} = t\, e^{\lambda_k t},\, \ldots,\, y_{k+m-1} = t^{m-1} e^{\lambda_k t}$$

```
(%i185) A: matrix([11,-25],[4,-9])$ Y: matrix([x],[y])$
        Yp: A.Y$ p: factor(charpoly(A,lambda));
```

(%o188) $(\lambda-1)^2$

```
(%i189) plotdf([Yp[1][1],Yp[2][1]],[trajectory_at,2,0],
        [x,-2,2],[y,-2,2],[direction,forward],
        [versus_t,1])$
(%i190)
```



$$\boldsymbol{y} = c_1\, e^t\, \boldsymbol{x}_1 + c_2\, t\, e^t\, \boldsymbol{x}_2$$