Overview

- Overview: nonlinear systems of differential equations
- Flow maps, stable and unstable equilibria
- Poincaré sections
- Examples:
  - Logistic map
  - Duffing oscillator
  - Lorenz system
  - Van der Pol oscillator

- Nonlinear systems of differential equations $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y})$, $\boldsymbol{y}: \mathbb{R} \to \mathbb{R}^n$, $\boldsymbol{f}: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$
- Very few nonlinear systems can be solved analytically
- Solutions can be found by numerical approximation (e.g., Euler, Runge-Kutta)
- Numerical solutions can be combined with analysis of qualitative behavior
- Tools:
  - $\boldsymbol{f}$ continuous $\Rightarrow$ solutions are unique, trajectories from different initial values do not cross
  - Effect of system parameters $\boldsymbol{p}$, $\boldsymbol{y}' = \boldsymbol{f}(t, \boldsymbol{y}; \boldsymbol{p})$, is crucial in understanding behavior, e.g.

$$my'' + cy' + ky = f, \boldsymbol{p} = \begin{pmatrix} m & c & k \end{pmatrix}$$

  - *Flow map*: $\Phi_t(\boldsymbol{y}_0; \boldsymbol{p}) = \boldsymbol{y}(t; \boldsymbol{y}_0; \boldsymbol{p})$ is the family of trajectories that start from initial condition $(0, \boldsymbol{y}_0)$. Interest is to determine how families of trajectories change when varying the system parameters $\boldsymbol{p}$

- Many of the features of dynamical system analysis are exhibited by a simple model, the logistic map describing the population $\tilde{y}(\tilde{t})$ of a species.
- Consider first the Malthusian model: population increase is proportional to current population

$$\tilde{y}' = r\,\tilde{y} \Rightarrow \tilde{y}(\tilde{t}) = e^{rt}\,\tilde{y}_0,$$

  that predicts (unbounded) exponential growth starting from initial population $\tilde{y}_0$.
- Modify the above model to include the effect of diminshing resources by modifying the growth rate to become $r(1-\tilde{y}/K)$. When the population $\tilde{y}$ reaches the carrying capacity K, the population growth rate becomes zero

$$\tilde{y}' = r\,\tilde{y}\left(1 - \frac{\tilde{y}}{K}\right) \Rightarrow \tilde{y}(\tilde{t}) = \frac{K e^{r\tilde{t}}}{K + y_0(e^{r\tilde{t}} - 1)}\,\tilde{y}_0$$

- Are two parameters needed? No, rescaling time $t = \tilde{t}/r$, and population $y = \tilde{y}/K$ leads to

$$y' = y\,(1-y),\, y(t) = \frac{e^t}{1 + y_0(e^t - 1)}\,y_0$$

  a differential equation with no parameters.

- A first concept in dynamical systems analysis is that of an *equilibrium point*, a state of the system that does not change. For the logistic map $y' = y(1-y)$ there are two equilibria

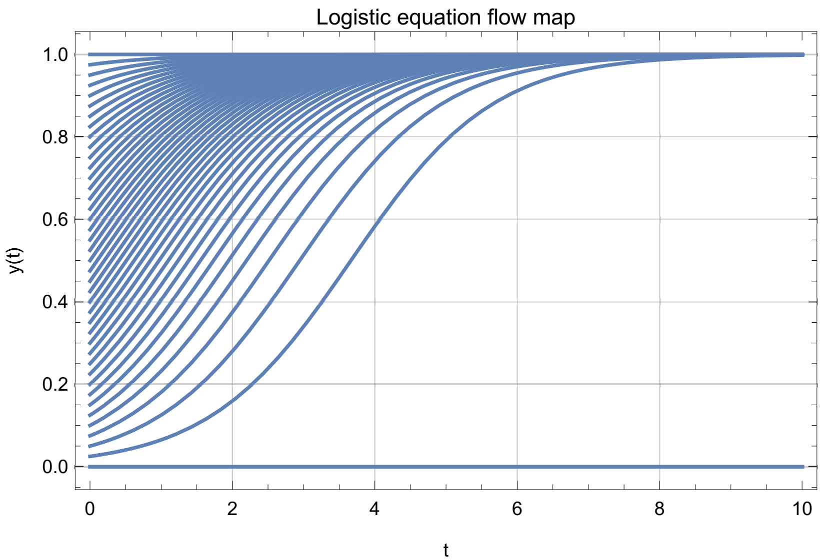
$$y = 0 \text{ and } y = 1.$$

- A second concept is that of *equilibrium point stability*, if the system is slightly displaced from an equilibrium point, does it return to its previous state or does it evolve to a different equilibrium?
- Consider $y' = f(y)$. Equilibria are determined by roots of f, $f(y) = 0$. Denote a root by y^*.
- For the logistic equation the equilibria are $y_1^* = 0$ and $y_2^* = 1$
- Asymptotic behavior of solution to logistic equation

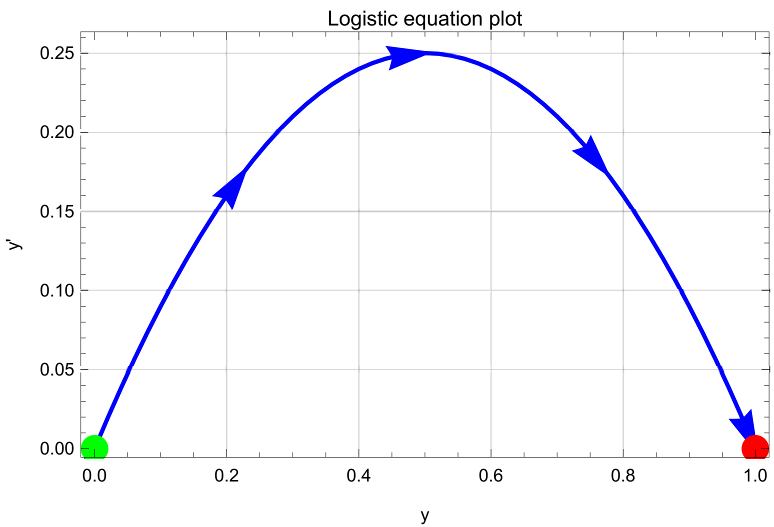
$$\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \frac{e^t}{1 + y_0(e^t - 1)} y_0 = 1 \text{ if } y_0 \neq 0$$

  - $y_1^* = 0$ is an *unstable equilibrium point*, after small perturbation $y \to 1 \neq y_1^*$
  - $y_2^* = 1$ is a *stable equilibrium point*, after small perturbation, $y \to 1 = y_2^*$

- The flow map for logistic equation is $\Phi_t(y_0) = y(t; y_0) = e^t\, y_0 / [1 + y_0(e^t - 1)]$
- Figure shows $\Phi_t(0), \Phi_t(0.05), \ldots$



- Same information is more economically shown in a phase plot of $(y(t), y'(t)) = (y, y(1-y))$
- A single curve showing flow from unstable equilibrium $(0, 0)$ to stable equilibrium $(1, 0)$
- Also: does not require knowledge of solution since $y' = f(y)$ is given



- Consider unforced harmonic oscillator $y'' + \gamma y' + \kappa y = 0$, $y' = v$, $v' = -\gamma v - \kappa y$
- Phase portrait (y', v') can readily be represented and the flow map $\Phi_t(\boldsymbol{y}_0; \boldsymbol{p}) = \boldsymbol{y}(t; \boldsymbol{y}_0; \boldsymbol{p})$ is known analytically, e.g., $\Phi_t(\boldsymbol{y}_0; \gamma = 0, \kappa) = (\cos\sqrt{\kappa}\, t, \sin\sqrt{\kappa}\, t)$.

```
(%i82) plotdf([v,-gamma*v-kappa*y],[y,v],
        [trajectory_at,.5,.5],
        [y,-1,1],[v,-1,1],
        [direction,forward],
        [parameters,"gamma=0,kappa=1"],
        [sliders,"gamma=0:2,kappa=0.5:2.0"])$
(%i83)
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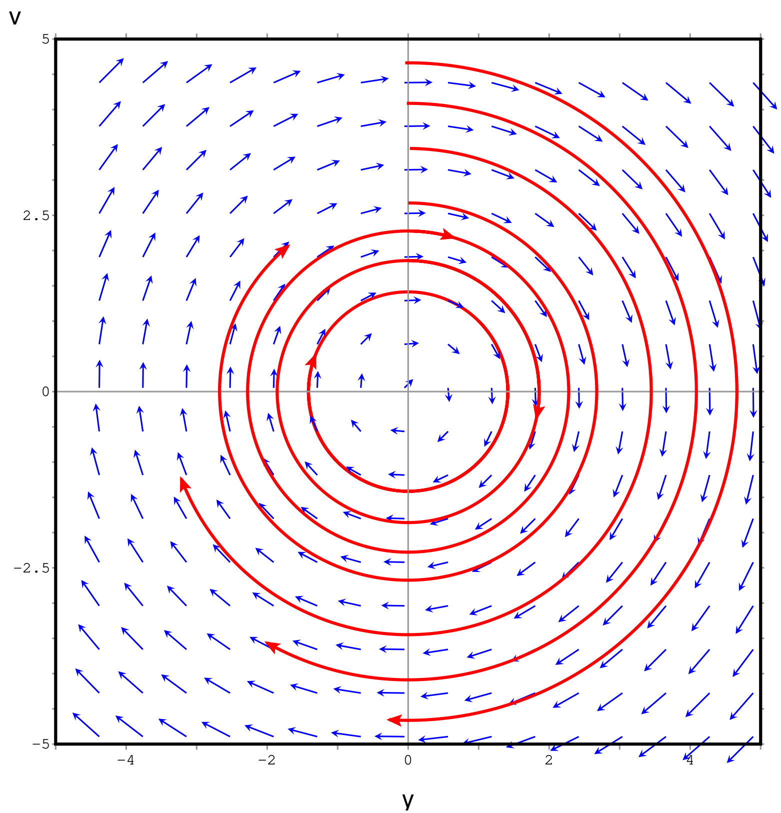
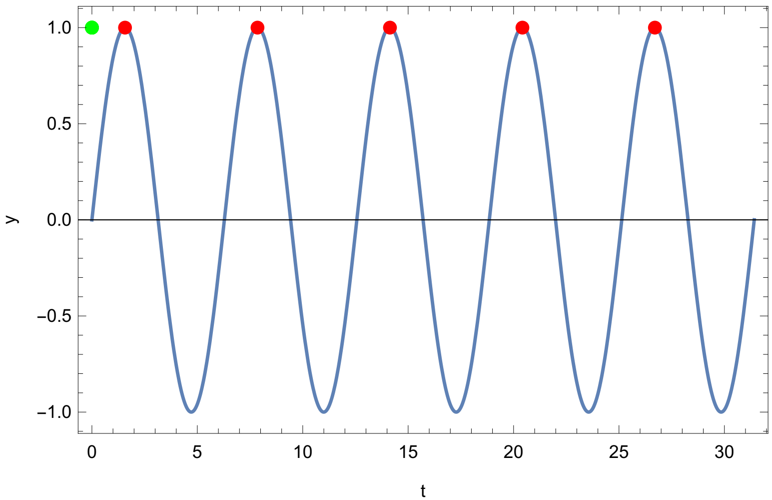


- However when $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y})$, $\boldsymbol{y} \in \mathbb{R}^n$, $n > 2$, phase portraits become difficult to represent graphically
- Observations:
  - for $\gamma = 0$, trajectories cross axes at regular intervals, at the same point
  - for $\gamma > 0$, trajectories cross axes at regular intervals, approaching origin as $t \to 0$

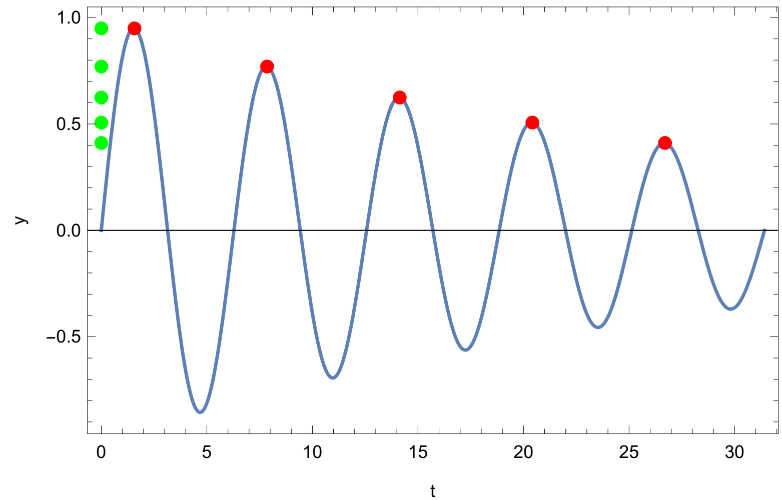
Poincaré sections

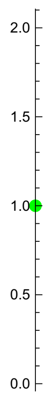
- The *stroboscopic effect* allows visualization of rotational or oscillatory motion

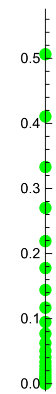
Undamped motion



Damped motion



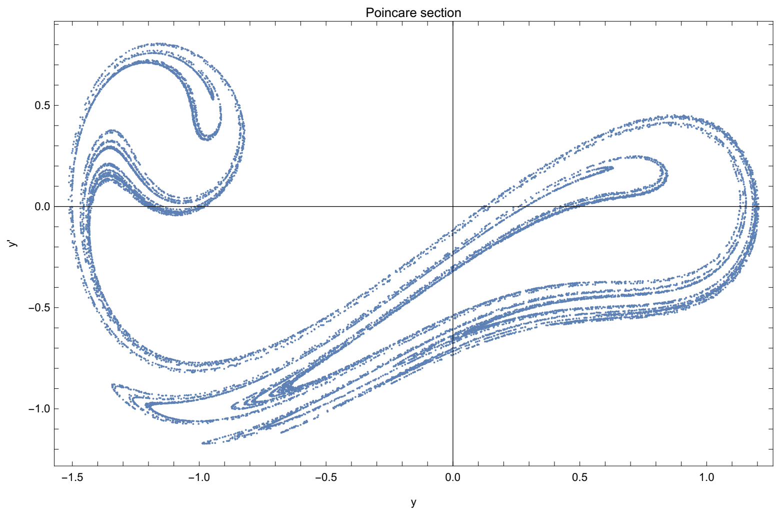


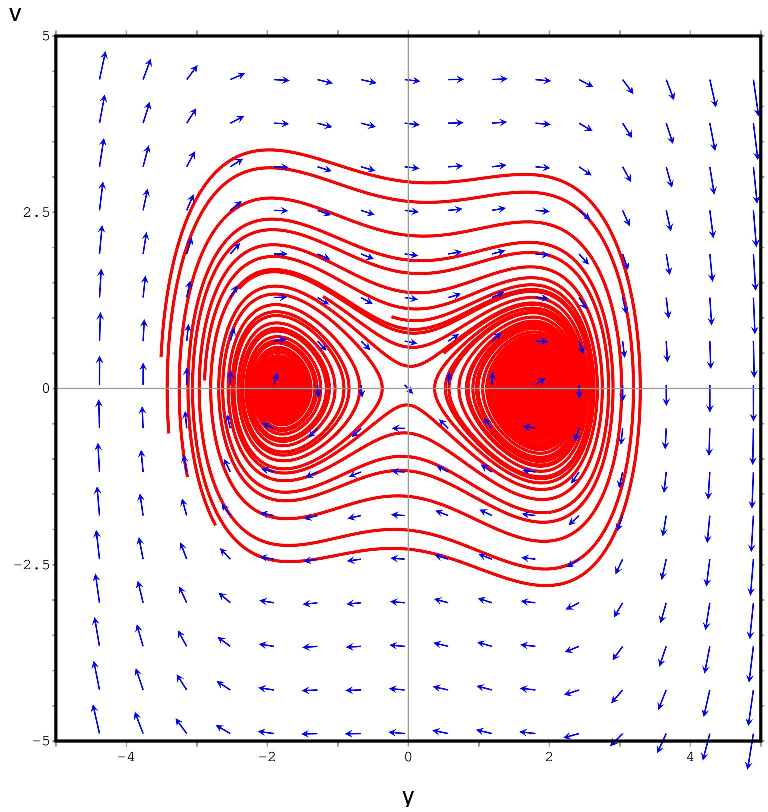


A Poincaré section is the projection onto the y-axis of points sampled at periodic intervals

- Consider the forced, non-linear oscillator, $y'' + \delta y' + \alpha y + \beta y^3 = \gamma \sin(\omega t)$, (Duffing)

```
(%i86) plotdf([v,-v/10+y-beta*y^3],[y,v],
        [trajectory_at,.5,.5],[nsteps,10000],
        [y,-5,5],[v,-5,5],
        [direction,forward],
        [parameters,"beta=0.25"],
        [sliders,"beta=0.1:0.4"])$
(%i87)
```

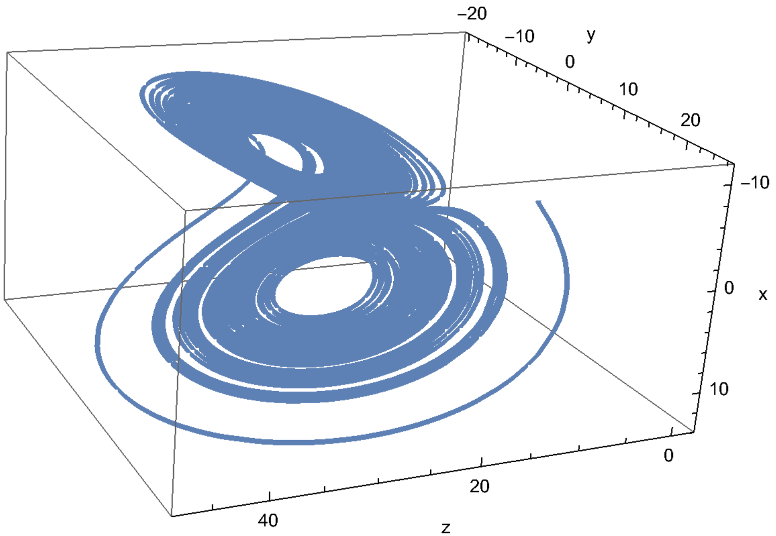




- E. Lorenz (1963, J. Atmos. Sci) "simple" model for weather prediction ($\beta, \rho, \sigma > 0$)

$$x' = \sigma\,(y - x)$$
$$y' = x(\rho - z) - y$$
$$z' = x\,y - \beta\,z$$

$$\boldsymbol{u}' = \boldsymbol{f}(\boldsymbol{u}), \boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

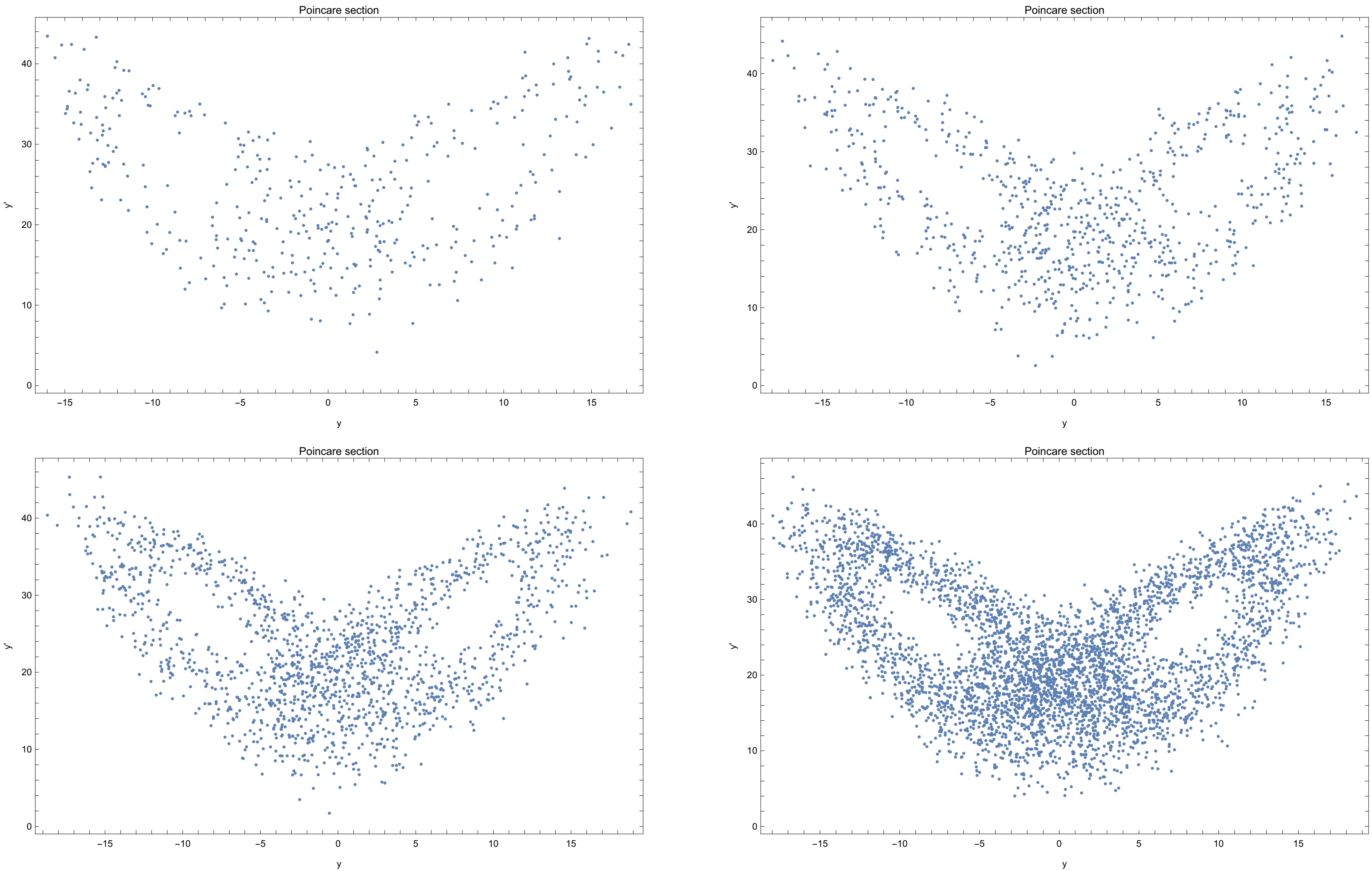


- Equilibria are solutions of $\boldsymbol{f}(\boldsymbol{u}) = \boldsymbol{0}$

$$\boldsymbol{u}_1^* = \boldsymbol{0}$$

$$\boldsymbol{u}_{2,3}^* = \begin{pmatrix} \pm\sqrt{\beta(\rho - 1)} \\ \pm\sqrt{\beta(\rho - 1)} \\ \rho - 1 \end{pmatrix}$$

Lorenz system Poincaré sections



- Successive construction of Poincaré sections, $m = 2500, 5000, 10000, 25000$ samples

- $x'' - \mu(1-x^2)x' + x = 0 \Rightarrow$

$$x' = y,\, y' = \mu(1-x^2)y - x$$

```
(%i4) plotdf([y,mu*(1-x^2)*y-x],[x,y],
      [trajectory_at,1,1],
      [x,-5,5],[y,-5,5],
      [direction,forward],
      [parameters,"mu=1"],[sliders,"mu=-1:1"],
      [versus_t,1])$
(%i5)
```