Overview

- Local linearization, eigenvalue evolution maps
- Forced systems, period doubling
- Chaotic behavior, implications for nonlinear systems: weather prediction, economic forecasts

- Behavior of solutions of nonlinear system $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{y} \in \mathbb{R}^n$ can be approximated locally by

$$\boldsymbol{y}' \cong \boldsymbol{f}_k + \boldsymbol{A}_k(\boldsymbol{y} - \boldsymbol{y}_k), \boldsymbol{f}_k = \boldsymbol{f}(\boldsymbol{y}_k), \boldsymbol{A}_k = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_k)$$

- In the vicinity of each point $\boldsymbol{y}_k$, eigenvalues $\lambda_{k,1}, \ldots, \lambda_{k,n}$ of $\boldsymbol{A}_k$ indicate localized behavior:
  - periodic orbits $\operatorname{Re} \lambda_{k,j} = 0$
  - attractor $\lambda_{k,j} < 0$ for all j
  - repeller $\lambda_{k,j} > 0$ for all j
  - saddle points $\lambda_{k,j} \in \mathbb{R}$ of differing signs

- Consider now behavior of solutions of forced nonlinear system $\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}) + \boldsymbol{F}(t)$, $\boldsymbol{y} \in \mathbb{R}^n$
- Local linearization is

$$\boldsymbol{y}' \cong \boldsymbol{f}_k + \boldsymbol{A}_k(\boldsymbol{y} - \boldsymbol{y}_k) + \boldsymbol{F}(t),\ \boldsymbol{f}_k = \boldsymbol{f}(\boldsymbol{y}_k),\ \boldsymbol{A}_k = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{y}_k)$$

- The behavior now depends on how each possible behavior is influenced by the force $\boldsymbol{F}(t)$. As in the unforced case, In the vicinity of each point $\boldsymbol{y}_k$, eigenvalues $\lambda_{k,1}, \ldots, \lambda_{k,n}$ of $\boldsymbol{A}_k$ indicate localized behavior, but can now be modified by $\boldsymbol{F}(t)$
  - periodic orbits $\operatorname{Re} \lambda_{k,j} = 0$, modified by the periods of $\boldsymbol{F}(t)$: resonance, period doubling
  - attractor $\lambda_{k,j} < 0$ for all j, modified by $\boldsymbol{F}(t)$: reinforce, counteract dissipation
  - repeller $\lambda_{k,j} > 0$ for all j, modified by $\boldsymbol{F}(t)$: system stabilization
  - saddle points $\lambda_{k,j} \in \mathbb{R}$ of differing signs, modified by $\boldsymbol{F}(t)$: maintain unstable system equilibrium
- See Homework12 solution for numerical experiments showing the above behaviors

- Systems can exhibit complex behavior when periods in the forcing and those intrinsic to the system are incomensurable
- Such behavior is recognized by complex Poincare sections, typically fractal, i.e., of non-integer dimension, and is termed "chaotic"
- Chaotic behavior arises surprisingly often, and is considered to be generic (i.e., expected), in the sense that a forced, nonlinear system will typically show chaotic behavior for some choice of system parameters
- Famous example: Lorenz system for weather prediction

  > *If the flap of a* butterfly's *wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado*. (E. Lorenz, 1972)

- Further study: Dynamical systems, Ergodic Theory (MATH590, MATH897)