Practice Test 1

Solve the following problems (3 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. Organize your computation and writing so the solution you present is readily legible. No credit is awarded for statement of the final answer to a problem without presentation of solution procedure.

1. Verify that

$$y = \begin{cases} e^x - 1, & x \geqslant 0, \\ 1 - e^{-x}, & x < 0, \end{cases}$$

is a solution of $y' = |y| + 1$.

Solution. For $x > 0$, $y(x) = e^x - 1 > 0$, which substituted in the differential equation $y' = y + 1$ gives $e^x = e^x - 1 + 1 \Rightarrow 0 = 0$, verified. For $x < 0$, $y(x) = 1 - e^{-x} < 0$, which substituted in the differential equation $y' = -y + 1$ gives $e^{-x} = -1 + e^{-x} + 1 \Rightarrow 0 = 0$, verified. Consider now the limiting behavior as $x \to 0$ of $y(x)$, and

$$y' = \begin{cases} e^x, & x > 0, \\ e^{-x}, & x < 0. \end{cases}$$

Since $\lim_{x \to 0, x < 0} y(x) = \lim_{x \to 0, x > 0} y(x) = y(0) = 0$, deduce that $y(x)$ is continuous everywhere, and a solution of $y' = |y| + 1$. Note that $\lim_{x \to 0, x < 0} y'(x) = \lim_{x \to, x > 0} y'(x) = 1 = |y(0)| + 1$, hence the derivative y' is also continuous everywhere and $y' = |y| + 1$ for all x, and the above is indeed a solution.

2. Use variation of parameters and separation of variables to solve

$$y' + y = \frac{2xe^{-x}}{1 + ye^x}.$$

Solution. Solve the homogeneous equation $y' + y = 0$ to obtain $y_h = e^{-x}$. By variation of parameters, assume solution is of form $y = uy_h$. Replace in above to obtain

$$(uy_h)' + (uy_h) = u'e^{-x} - ue^{-x} + ue^{-x} = u'e^{-x} = \frac{2xe^{-x}}{1 + u} \Rightarrow (1 + u)u' = 2x.$$

The differential equation in u is separable and a solution is found by integration

$$\int (1 + u)\,\mathrm{d}u = 2\int x\,\mathrm{d}x \Rightarrow u + \frac{u^2}{2} = x^2 + c \Rightarrow u^2 + 2u - 2(x^2 + c) = 0.$$

Solutions of the quadratic equation are

$$u = -1 \pm (1 + 2(x^2 + c))^{1/2},$$

so the differential equation has two solutions $y = e^{-x}[-1 \pm (1 + 2(x^2 + c))^{1/2}]$

3. Find all (x_0, y_0) for which the initial value problem

$$y' = \frac{\tan y}{x - 1}, y(x_0) = y_0$$

has a unique solution on some open interval that contains x_0.

Solution. For the IVP to have a unique solution $f(x, y) = \tan y / (x - 1)$ must be continuous with a continuous derivative in y, $f_y = \sec^2 y / (x - 1) = 1 / [(x - 1)\cos^2 y]$. Both f and f_y are discontinuous at when $\cos y = 0 \Rightarrow y = (2k + 1)\pi / 2, k \in \mathbb{Z}$. Furthermore f is also discontinuous at $x = 0$. For the IVP to have an unique solution for some open interval that contains x_0, we must have $x_0 \neq 0$ and $y \neq (2k + 1)\pi, k \in \mathbb{Z}$.

4. Find all functions M such that $M(x, y)\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y = 0$ is exact.

Solution. The differential $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{dy} = 0$ is exact if $M_y = N_x$, or

$$M_y = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x \Rightarrow M = 2xy + f(x),$$

with $f(x)$ some arbitrary (differentiable) function of x.