Practice Test 3

Solve the following problems (3 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. Organize your computation and writing so the solution you present is readily legible. No credit is awarded for statement of the final answer to a problem without presentation of solution procedure.

This is an open-book test, and you are free to consult the textbook or use software to check your solution. Note however that the questions are so formulated that it is more efficient to draft the solution without use of software or consultation of the textbook; both of those actions would rapidly use up the allotted time. If you studied the course material and understood solutions to the homework assignments, drafting test question solutions in TeXmacs should take about 90 minutes. The allotted time is 3 hours for everyone, thus also providing special needs accomodation.

Draft your solution in TeXmacs. At least 10 minutes before the submission cut-off time, copy and paste your answer into Sakai. Remember to use Edit->Copy to->TeXmacs. See the webinar for an example.

1. Rewrite the initial value problem in matrix form and find a fundamental solution set of solutions

$$\begin{cases} y_1' = 6y_1 + 4y_2 + 4y_3, & y_1(0) = 3, \\ y_2' = -7y_1 - 2y_2 - y_3, & y_2(0) = -6, \quad (4a520) \\ y_3' = 7y_1 + 4y_2 + 3y_3, & y_3(0) = 4. \end{cases}$$

Solution. In matrix form

$$\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \boldsymbol{Y}' = \boldsymbol{A}\boldsymbol{Y}, \boldsymbol{A} = \begin{pmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{pmatrix}.$$

Find the eigenvalues of

$$\boldsymbol{A} = \begin{pmatrix} 6 & 4 & 4 \\ -7 & -2 & -1 \\ 7 & 4 & 3 \end{pmatrix}; p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda-6 & -4 & -4 \\ 7 & \lambda+2 & 1 \\ -7 & -4 & \lambda-3 \end{vmatrix} \Rightarrow$$

$$p(\lambda) = \begin{vmatrix} \lambda-6 & -4 & -4 \\ 0 & \lambda-2 & \lambda-2 \\ -7 & -4 & \lambda-3 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-6 & -4 & -4 \\ 0 & 1 & 1 \\ -7 & -4 & \lambda-3 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda-6 & -4 & 0 \\ 0 & 1 & 0 \\ -7 & -4 & \lambda+1 \end{vmatrix} \Rightarrow$$

$$p(\lambda) = (\lambda-2)\begin{vmatrix} \lambda-6 & 0 \\ -7 & \lambda+1 \end{vmatrix} = (\lambda-2)(\lambda-6)(\lambda+1)$$

and the fundamental set of solutions is

$$\{e^{2t}, e^{6t}, e^{-t}\}$$

2. Solve the initial value problem (4a520).

First find the eigenvectors of $\boldsymbol{A}$ by row reduction. For $\lambda = -1$

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} 7 & 4 & 4 \\ -7 & -1 & -1 \\ 7 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 7 & 4 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 7 & 4 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Similar procedures for $\lambda = 2$, $\lambda = 6$ give

$$\boldsymbol{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \boldsymbol{x}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The general solution is

$$\boldsymbol{y} = c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{6t}$$

At $t = 0$,

$$c_1 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -6 \\ 4 \end{pmatrix}$$

with solution $c_1 = 1$, $c_2 = 1$, $c_3 = 2$.

3. Find a fundamental set of solutions for the system of differential equations

$$\boldsymbol{y}' = \begin{pmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{pmatrix} \boldsymbol{y} \; (24.556)$$

Solution. Find the eigenvalues of

$$\boldsymbol{A} = \begin{pmatrix} 5 & -1 & 1 \\ -1 & 9 & -3 \\ -2 & 2 & 4 \end{pmatrix}; p(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \left| \begin{pmatrix} \lambda - 5 & 1 & -1 \\ 1 & \lambda - 9 & 3 \\ 2 & -2 & \lambda - 4 \end{pmatrix} \right| \Rightarrow$$

$$p(\lambda) = \begin{vmatrix} \lambda - 5 & 1 & -1 \\ 1 & \lambda - 9 & 3 \\ 2 & -2 & \lambda - 4 \end{vmatrix} = \begin{vmatrix} \lambda - 4 & 1 & -1 \\ \lambda - 8 & \lambda - 9 & 3 \\ 0 & -2 & \lambda - 4 \end{vmatrix} = \begin{vmatrix} \lambda - 4 & 0 & -1 \\ \lambda - 8 & \lambda - 6 & 3 \\ 0 & \lambda - 6 & \lambda - 4 \end{vmatrix} = (\lambda - 6) \begin{vmatrix} \lambda - 4 & 0 & -1 \\ \lambda - 8 & 1 & 3 \\ 0 & 1 & \lambda - 4 \end{vmatrix} \Rightarrow$$

$$p(\lambda) = (\lambda - 6) \begin{vmatrix} \lambda - 4 & 0 & -1 \\ \lambda - 8 & 0 & 3 - (\lambda - 4) \\ 0 & 1 & \lambda - 4 \end{vmatrix} = (\lambda - 6) \begin{vmatrix} \lambda - 4 & 0 & -1 \\ \lambda - 8 & 0 & 7 - \lambda \\ 0 & 1 & \lambda - 4 \end{vmatrix} = -(\lambda - 6) \begin{vmatrix} \lambda - 4 & -1 \\ \lambda - 8 & 7 - \lambda \end{vmatrix} \Rightarrow$$

$$p(\lambda) = -(\lambda - 6)[(\lambda - 4)(7 - \lambda) + \lambda - 8] = -(\lambda - 6)(-\lambda^2 + 12\lambda - 36) = (\lambda - 6)^3.$$

The fundamental set of solutions for this triple root case is

$$e^{6t}, t e^{6t}, t^2 e^{6t}.$$

4. Find a fundamental set of solutions for the system of differential equations

$$\boldsymbol{y}' = \begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{pmatrix} \boldsymbol{y} \; (5.567)$$

Solution.

$$\boldsymbol{A}=\begin{pmatrix} 3 & -3 & 1 \\ 0 & 2 & 2 \\ 5 & 1 & 1 \end{pmatrix}; p(\lambda)=\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix} \lambda-3 & 3 & -1 \\ 0 & \lambda-2 & -2 \\ -5 & -1 & \lambda-1 \end{vmatrix} \Rightarrow$$

$$p(\lambda)=\begin{vmatrix} \lambda-3 & 3 & -1 \\ 0 & \lambda-2 & -2 \\ -5 & -1 & \lambda-1 \end{vmatrix}=\begin{vmatrix} \lambda+2 & 4 & -\lambda \\ 0 & \lambda-2 & -2 \\ -5 & -1 & \lambda-1 \end{vmatrix}=\begin{vmatrix} \lambda+2 & \lambda+2 & -(\lambda+2) \\ 0 & \lambda-2 & -2 \\ -5 & -1 & \lambda-1 \end{vmatrix} \Rightarrow$$

$$p(\lambda)=(\lambda+2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & \lambda-2 & -2 \\ -5 & -1 & \lambda-1 \end{vmatrix}=(\lambda+2)\begin{vmatrix} 1 & 0 & 0 \\ 0 & \lambda-2 & -2 \\ -5 & 4 & \lambda-6 \end{vmatrix}=(\lambda+2)\begin{vmatrix} \lambda-2 & -2 \\ 4 & \lambda-6 \end{vmatrix} \Rightarrow$$

$$p(\lambda)=(\lambda+2)[(\lambda-2)(\lambda-6)+8]=(\lambda+2)(\lambda^2-8\lambda+20).$$

The fundamental set of solutions for this case

$$e^{-2t}, e^{4t}\cos 2t, e^{4t}\sin(2t).$$