# Homework 2

Due date: Jan 23, 2019, 11:55PM.

Bibliography: Trench Chap. 2

1. Exercises 6-9, p. 41

**Ex 6.** $y' + (1+x)y/x = 0$, $y(1) = 1$. Assume $x \neq 0$, and solve by integration

$$\frac{y'}{y} = -\frac{x+1}{x} \Rightarrow \int \frac{\mathrm{d}y}{y} = -\int \frac{x+1}{x}\,\mathrm{d}x \Rightarrow \ln y - \ln c = -(x + \ln x) \Rightarrow y = \frac{c}{x} e^{-x}. y(1) = \frac{c}{e} = 1 \Rightarrow c = e$$

Solution is $y = e^{1-x}/x$. Check in Maxima

```
(%i4) ode2('diff(y,x)/y=-(x+1)/x,y,x);
```

(%o4) $y = \frac{\%c\, \mathrm{e}^{-x}}{x}$

```
(%i5) ic1(%,x=1,y=1);
```

(%o5) $y = \frac{\mathrm{e}^{1-x}}{x}$

**Ex 7.** $x y' + (1 + 1/\ln x) y = 0, y(e) = 1$. Assume $x \neq 0$, and solve by integration

$$\frac{y'}{y} = -\frac{1}{x \ln x} - \frac{1}{x} \Rightarrow \ln y - \ln c = -\int \left(\frac{1}{x \ln x} + \frac{1}{x}\right) \mathrm{d}x = -\ln(\ln x) - \ln x = -\ln(x \ln x) \Rightarrow y = \frac{c}{x \ln x}$$

$$y(e) = \frac{c}{e} = 1 \Rightarrow y = \frac{e}{x \ln x}.$$

Verify in Maxima:

```
(%i9) ode2('diff(y,x)*x+(1+1/log(x))*y=0,y,x);
```

(%o9) $y = \frac{\%c}{x \log(x)}$

```
(%i10) ic1(%,x=%e,y=1);
```

(%o10) $y = \frac{\mathrm{e}}{x \log(x)}$

```
(%i11)
```

**Ex 8.** $x y' + (1 + x \cot x)\, y = 0, y(\pi/2) = 2$. Assume $x \neq 0$, and solve by integration

$$\frac{y'}{y} = -\frac{1}{x} - \frac{1}{\cot x} \Rightarrow \ln y - \ln c = -\ln x - \ln \sin x \Rightarrow y = \frac{c}{x \sin x}$$

$$y(\pi/2) = 2c/\pi = 2 \Rightarrow c = \pi.\ y = \frac{\pi}{x \sin x}$$

Verify in Maxima

```
(%i16) ode2('diff(y,x)*x+(1+x*cot(x))*y=0,y,x);
```

(%o16) $y = \frac{\%c}{x \sin(x)}$

(%i17) `ic1(%,x=%pi/2,y=2);`

(%o17) $y = \frac{\pi}{x \sin(x)}$

(%i18)

**Ex 9.** $y' + ky/x = 0$, $y(1) = 3$. Integrate:

$$\frac{y'}{y} = -\frac{k}{x} \Rightarrow \ln y - \ln c = -k \ln x \Rightarrow y = cx^{-k}, y(1) = c = 3. y(x) = 3x^{-k}$$

(%i22) `ode2('diff(y,x)+k*y/x=0,y,x);`

(%o22) $y = \%c\, \mathrm{e}^{-k \log(x)}$

(%i23) `ic1(%,x=1,y=3);`

(%o23) $y = 3\, \mathrm{e}^{-k \log(x)}$

(%i24)

2. Exercises 12-15, p.41

**Ex 12.** $y' + 3y = 1$. Solve by integration

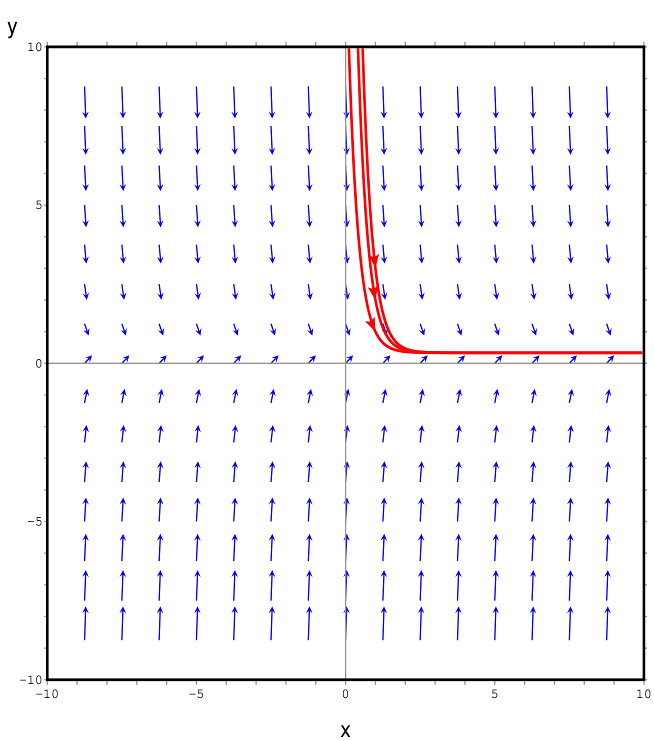
$$y' = 1 - 3y = \frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \mathrm{d}x = \frac{\mathrm{d}y}{1 - 3y} \Rightarrow x + c = -\frac{1}{3} \ln(1 - 3y)$$

(%i1) `ode2('diff(y,x)=1-3*y,y,x);`

(%o1) $y = \mathrm{e}^{-3x} \left( \frac{\mathrm{e}^{3x}}{3} + \%c \right)$

(%i3) `plotdf(1-3*y);`

(%i4)



**Ex 13.** $y' + (1/x - 1)y = -2/x$. Use variation of parameters. Homogeneous equation $y' + (1/x - 1)y = 0$ has solution:

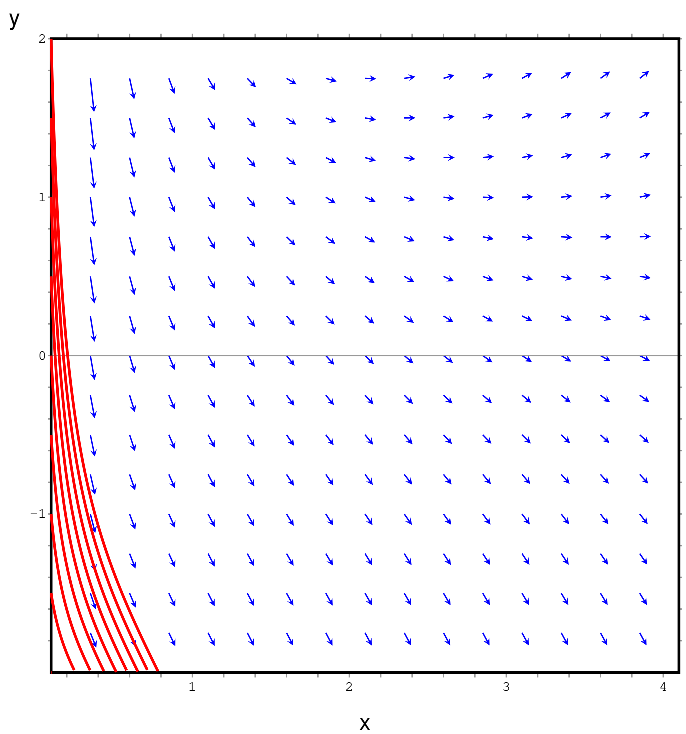
$$\frac{y'}{y} = 1 - \frac{1}{x} \Rightarrow \ln y - \ln c = x - \ln x \Rightarrow y_h = e^x/x$$

Seek solution of form $y = u y_h \Rightarrow u' = -2e^{-x} \Rightarrow u = 2e^{-x} + c \Rightarrow y = (2 + ce^x)/x$. Verify in Maxima

```
(%i3) ode2('diff(y,x)+(1/x-1)*y=-2/x,y,x);
```

(%o3) $y = \frac{(2\,e^{-x} + \%c)\,e^x}{x}$

```
(%i5) plotdf(-2/x-(1/x-1)*y,[x,.1,4.1],[y,-2,2]);
```



(%i6)

**Ex 14.** $y'+2xy=xe^{-x^2}$. Solve by variation of parameters. Homogeneous equation solution

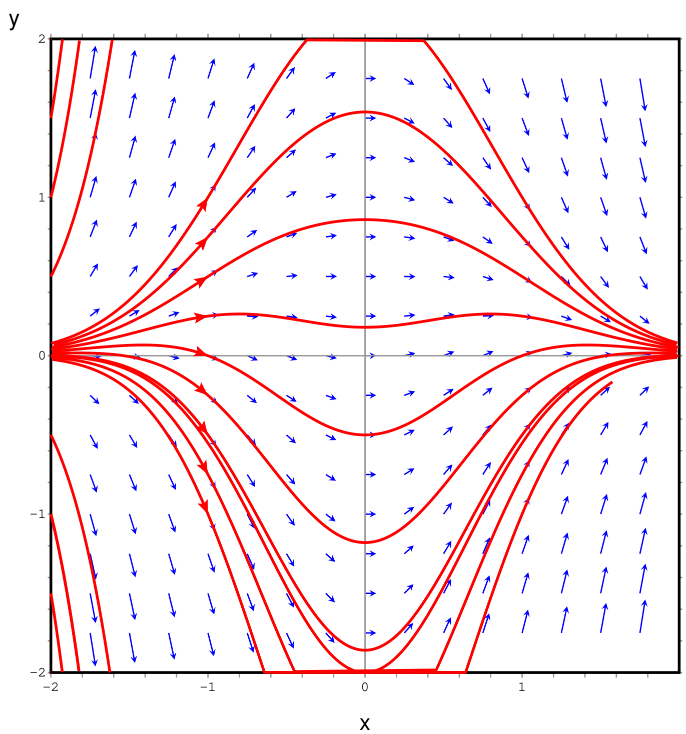
$$\frac{y'}{y}=-2x\Rightarrow y_h=e^{-x^2}\,(\text{particular solution})$$

Variation of parameters $y=u\,y_h\Rightarrow u'=x\Rightarrow u=x^2/2+c\Rightarrow y=(x^2/2+c)e^{-x^2}$. Verify in Maxima

```
(%i7) ode2('diff(y,x)+2*x*y=x*exp(-x^2),y,x);
```

(%o7) $y=\left(\frac{x^2}{2}+\%c\right)\mathrm{e}^{-x^2}$

```
(%i9) plotdf(-2*x*y+x*exp(-x^2),[x,-2,2],[y,-2,2]);
```




**Ex 15.** $y'+2xy/(1+x^2)=e^{-x}/(1+x^2)$. Solve by variation of parameters. Homogeneous solution:

$$\frac{y'}{y}=-\frac{2x}{1+x^2}\Rightarrow y_h=\frac{1}{1+x^2}.$$

Seek solution $y=u\,y_h\Rightarrow$

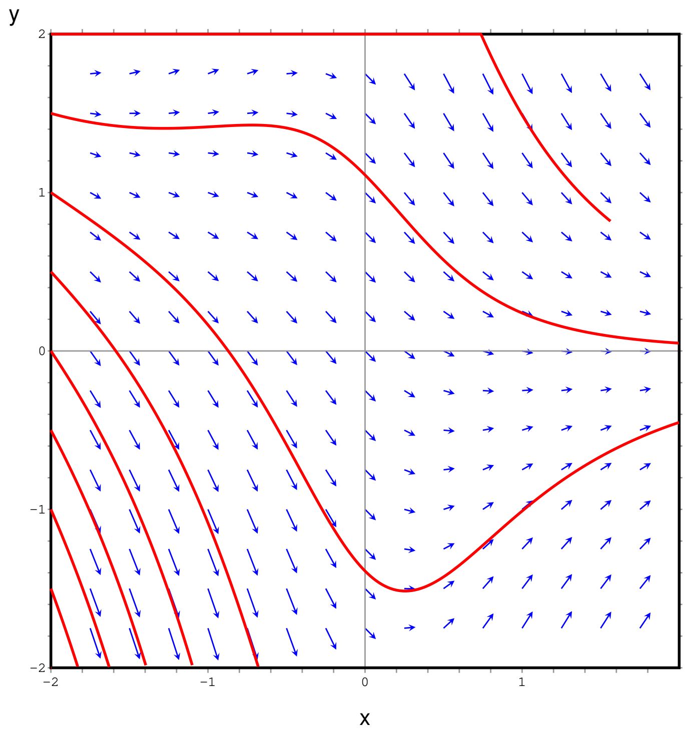
$$u'=e^{-x}\Rightarrow u=-e^{-x}+c\Rightarrow y=\frac{c-e^{-x}}{1+x^2}$$

Verify in Maxima

```
(%i14) ode2('diff(y,x)+2*x*y/(1+x^2)=exp(-x)/(1+x^2),y,x);
```

(%o14) $y=\frac{\%c-\mathrm{e}^{-x}}{x^2+1}$

```
(%i15) plotdf(-2*x*y/(1+x^2)-exp(-x)/(1+x^2),[x,-2,2],[y,-2,2]);
```




```
(%i16)
```

3. Exercises 30-33, p.42

**Ex 30.**

$$(x-1)y' + 3y = \frac{1}{(x-1)^3} + \frac{\sin x}{(x-1)^2},\, y(0) = 1.$$

Solve by variation of parameters. Homogeneous solution

$$\frac{y'}{y} = -\frac{3}{x-1} \Rightarrow \ln y = -3\ln(x-1) \Rightarrow y_h = \frac{1}{(x-1)^3}.$$

Seek solution as $y = u y_h$.

$$u' = \frac{1}{x-1} + \sin x \Rightarrow u = \ln(x-1) - \cos x + c \Rightarrow$$

$$y = \frac{\ln(x-1) - \cos x + c}{(x-1)^3},\, y(0) = 1 - c = 1 \Rightarrow c = 0$$

Verify

```
(%i6) y30: (log(x-1)-cos(x))/(x-1)^3
```

(%o6) $\frac{\log(x-1) - \cos(x)}{(x-1)^3}$

```
(%i11) fullratsimp((x-1)*diff(y30,x)+3*y30);
```

(%o11) $\frac{(x-1)\sin(x) + 1}{x^3 - 3x^2 + 3x - 1}$

```
(%i12) partfrac(%,x);
```

(%o12) $\frac{\sin(x)}{(x-1)^2} + \frac{1}{(x-1)^3}$

(%i13)

**Ex 31.** $xy'+2y=8x^2$, $y(1)=3$. Solve by variation of parameters. Homogeneous solution:

$$\frac{y'}{y}=-\frac{2}{x}\Rightarrow y_h=x^{-2}$$

Try $y=uy_h\Rightarrow u'=8x^3\Rightarrow u=2x^4+c\Rightarrow y=(2x^2+c/x^2)$. Initial condition $y(1)=2+c=3\Rightarrow c=1$, and solution is $y=2x^2+c/x^2$. Verify in Maxima

```
(%i13) ode2('diff(y,x)*x+2*y=8*x^2,y,x);
```

(%o13) $y=\frac{2x^4+\%c}{x^2}$

```
(%i14) ic1(%,x=1,y=3);
```

(%o14) $y=\frac{2x^4+1}{x^2}$

```
(%i15) partfrac(%,x);
```

(%o15) $y=2x^2+\frac{1}{x^2}$

(%i16)

**Ex 32.** $xy'-2y=-x^2$, $y(1)=1$. Solve by variation of parameters. Homogeneous solution

$$\frac{y'}{y}=\frac{2}{x}\Rightarrow y_h=x^2.$$

Try $y=uy_h\Rightarrow u'=-1/x\Rightarrow u=-\ln x+c\Rightarrow y=(c-\ln x)\,x^2$. Initial condition: $y(1)=c=1$. Solution:

$$y=(1-\ln x)x^2.$$

Verify:

```
(%i16) ode2('diff(y,x)*x-2*y=-x^2,y,x);
```

(%o16) $y=x^2\,(\%c-\log(x))$

```
(%i17) ic1(%,x=1,y=1);
```

(%o17) $y=x^2-x^2\log(x)$

(%i18)

**Ex 33.** $y'+2xy=x$, $y(0)=3$. By variation of parameters. Homogeneous solution:

$$\frac{y'}{y}=-2x\Rightarrow y_h=e^{-x^2}.$$

Try $y=uy_h\Rightarrow u'=xe^{x^2}\Rightarrow u=\frac{1}{2}e^{x^2}+c\Rightarrow y=\left(\frac{1}{2}+ce^{-x^2}\right)$, $y(0)=\frac{1}{2}+c=3\Rightarrow c=\frac{5}{2}\Rightarrow$

$$y=\frac{1}{2}(1+5e^{-x^2}).$$

Verify:

```
(%i18) ode2('diff(y,x)+2*x*y=x,y,x);
```

(%o18) $y=\mathrm{e}^{-x^2}\left(\frac{\mathrm{e}^{x^2}}{2}+\%c\right)$

```
(%i19) ic1(%,x=0,y=3);
```

(%o19) $y=\frac{\mathrm{e}^{-x^2}\,(\mathrm{e}^{x^2}+5)}{2}$

```
(%i20)
```

4. Exercises 1-4, p. 52

**Ex 1.** $y' = (3x^2 + 2x + 1)/(y - 2)$. Assume $y \neq 2$, and integrate

$$\int (y - 2)\,\mathrm{d}y = \int (3x^2 + 2x + 1)\,\mathrm{d}x \Rightarrow \frac{1}{2}(y - 2)^2 = x^3 + x^2 + x + c.$$

There are two solutions

$$y = 2 \pm \sqrt{2(x^3 + x^2 + x + c)}$$

**Ex 2.** $(\sin x)(\sin y) + (\cos y)\,y' = 0$. Assume $\sin y \neq 0$, divide by $\sin y$ and integrate

$$\int \cot y\,\mathrm{d}y = -\int \sin x\,\mathrm{d}x \Rightarrow \ln(\sin y) = \cos x + c \Rightarrow \sin y = e^{c + \cos x}.$$

Solutions will only exist in intervals such that $0 < e^{c + \cos x} \leqslant 1 \Rightarrow c + \cos x \leqslant 0$. In such intervals there are infinitely many solutions $y = (-1)^k \arcsin e^{c + \cos x} + k\pi, k \in \mathbb{Z}$. Rewriting the DE as $y' = f(x, y) = -\sin x \tan y$, notice that $f$ is not continuous at $\pm k\pi/2$.

**Ex 3.** $x\,y' + y^2 + y = 0$. For $x \neq 0$, rewrite to obtain a Bernoulli equation $y' + y/x = -y^2/x$. Solve homogeneous equation

$$\frac{y'}{y} = -\frac{1}{x} \Rightarrow y_h = \frac{1}{x}.$$

Try $y = u\,y_h \Rightarrow u'\,y_h = -(u\,y_h)^2/x \Rightarrow$

$$\frac{u'}{u^2} = -\frac{y_h}{x} = -\frac{1}{x^2} \Rightarrow \frac{1}{u} = c - \frac{1}{x} \Rightarrow u = \frac{x}{cx - 1} \Rightarrow y = \frac{1}{cx - 1}$$

```
(%i23) y3: 1/(c*x-1);
```

(%o23) $\frac{1}{c\,x - 1}$

```
(%i24) x*diff(y3,x)+y3^2+y3
```

(%o24) $\frac{1}{c\,x - 1} - \frac{c\,x}{(c\,x - 1)^2} + \frac{1}{(c\,x - 1)^2}$

```
(%i25) fullratsimp(%)
```

(%o25) $0$

```
(%i26)
```

**Ex 4.** $y' \ln|y| + x^2\,y = 0$. For $y > 0$ obtain

$$\frac{y' \ln y}{y} = -x^2 \Rightarrow \frac{1}{2}(\ln y)^2 = -\frac{1}{3}x^3 + \frac{1}{2}c$$

There are two solutions

$$y = \exp\left[\pm\sqrt{c - \frac{2}{3}x^3}\right].$$