Homework 3

Due date: Jan 30, 2019, 11:55PM.

Bibliography: Trench Chap. 2

1. Exercises 1-4, p. 61

Ex.1. $y' = f(x, y) = (x^2 + y^2)/\sin x$. Both f and $f_y = 2y/\sin x$ are continous in y except when $x_0 = k\pi$. A unique solution is found for some interbal for any $x_0 \neq k\pi$, $k \in \mathbb{Z}$, any y_0.

Ex.2. $y' = f(x, y) = (e^x + y)/(x^2 + y^2)$. Calculate

$$f_y = \frac{x^2 + y^2 - 2(e^x + y)y}{(x^2 + y^2)^2}.$$

A unique solution is found for $(x_0, y_0) \neq (0, 0)$.

Ex.3. $y' = f(x, y) = \tan(xy)$. Compute

$$f_y = \frac{x}{\cos^2(xy)}$$

A unique solution is found for $\cos(x_0\, y_0) \neq 0 \Rightarrow x_0\, y_0 \neq k\pi + \pi/2$, $k \in \mathbb{Z}$.

Ex. 4. $y' = f(x, y) = (x^2 + y^2)/\ln(xy)$. Compute

$$f_y = \frac{2y\ln(xy) - \frac{x^2 + y^2}{y}}{\ln^2(xy)} = \frac{2y^2\ln(xy) - x^2 - y^2}{y\ln^2(xy)}.$$

A unique solution is found for $x_0 y_0 \neq 1$, $x_0 y_0 > 0$.

2. Exercises 15,16, p.61

Ex.15.

a) Compute y' by differentiating the analytical forms on each branch

$$y = \begin{cases} 0 & -\infty < x \leqslant -1 \\ (x^2 - 1)^{5/3} & -1 < x < 1 \\ 0 & 1 \leqslant x < \infty \end{cases} \Rightarrow y' = \begin{cases} 0 & -\infty < x < -1 \\ \frac{10x}{3}(x^2 - 1)^{2/3} & -1 < x < 1 \\ 0 & 1 < x < \infty \end{cases}.$$

At the transition point between branches the definition of the derivative from each side must be used

$$y_l'(-1) = \lim_{x \to -1, x < -1} \frac{y(x) - y(-1)}{x + 1} = \lim_{x \to -1, x < -1} \frac{0}{x + 1} = 0.$$

$$y_r'(-1) = \lim_{x \to -1, x > -1} \frac{y(x) - y(-1)}{x + 1} = \lim_{x \to -1, x > -1} \frac{(x^2 - 1)^{5/3}}{x + 1} = \lim_{x \to -1, x > -1} \frac{(x - 1)^{5/3}(x + 1)^{5/3}}{x + 1} = \lim_{x \to -1, x > -1} (x - 1)^{5/3}(x + 1)^{2/3} = 0.$$

Since $y_l'(-1) = y_r'(-1) = 0$, the derivative at $x = -1$ is well defined and $y'(-1) = 0$. A similar calculation gives $y'(1) = 0$, so the overall derivative can now be defined as

$$y' = \begin{cases} 0 & -\infty < x \leqslant -1 \\ \dfrac{10x}{3}(x^2-1)^{2/3} & -1 < x < 1 \\ 0 & 1 \leqslant x < \infty \end{cases}.$$

For $|x| \geqslant 1$, $y = 0$ implies $y' = 0$ and hence verifies $y' = 10xy^{2/5}/3$. For $|x| < 1$, $y = (x^2-1)^{5/3}$ implies

$$y' = \frac{10x}{3}(x^2-1)^{2/3},$$

and replacing in DE gives

$$\frac{10x}{3}(x^2-1)^{2/3} = \frac{10}{3}x\,[(x^2-1)^{5/3}]^{2/5} = \frac{10}{3}x\,(x^2-1)^{2/3}, \text{ verified.}$$

b) Compute y' by differentiating the analytical forms on each branch

$$y' = \begin{cases} \epsilon_1\,(x^2-a^2)^{5/3} & -\infty < x < -a \\ 0 & -a \leqslant x \leqslant -1 \\ (x^2-1)^{5/3} & -1 < x < 1 \\ 0 & 1 \leqslant x \leqslant b \\ \epsilon_2\,(x^2-b^2)^{5/3} & b < x < \infty \end{cases} \Rightarrow y' = \begin{cases} \epsilon_1 \dfrac{10x}{3}(x^2-a^2)^{2/3} & -\infty < x < -a \\ 0 & -a < x < -1 \\ \dfrac{10x}{3}(x^2-1)^{2/3} & -1 < x < 1 \\ 0 & 1 < x < b \\ \epsilon_2 \dfrac{10x}{3}(x^2-b^2)^{2/3} & b < x < \infty \end{cases}.$$

The derivative at the transition points between branches must be evaluated by computing limits. The case $\epsilon_1 = \epsilon_2 = 0$ is already proved in (a). When $\epsilon_1 = 1$ compute limits

$$y_l'(-a) = \lim_{x\to -a, x<-a} \frac{y(x)-y(-a)}{x+a} = \lim_{x\to -a, x<-a} \frac{(x^2-a^2)^{5/3}}{x+a} = \lim_{x\to -a, x<-a} \frac{(x-a)^{5/3}(x+a)^{5/3}}{x+a} =$$
$$\lim_{x\to -a, x<-a} (x-a)^{5/3}(x+a)^{2/3} = 0.$$

$$y_r'(-a) = \lim_{x\to -a, x>-a} \frac{y(x)-y(-1)}{x+a} = \lim_{x\to -a, x>-a} \frac{0}{x+a} = 0.$$

Calculation at $x = b$ for $\epsilon_2 = 1$ is similar, and the derivative is

$$y' = \begin{cases} \epsilon_1 \dfrac{10x}{3}(x^2-a^2)^{2/3} & -\infty < x < -a \\ 0 & -a \leqslant x \leqslant -1 \\ \dfrac{10x}{3}(x^2-1)^{2/3} & -1 < x < 1 \\ 0 & 1 \leqslant x \leqslant b \\ \epsilon_2 \dfrac{10x}{3}(x^2-b^2)^{2/3} & b < x < \infty \end{cases}.$$

and verifies the ODE on all branches.

3. Exercises 17,18, p.62

Ex. 17. $y' = f(x, y) = 3x(y-1)^{1/3}$. (a) Since f is continuous in y for $y \geqslant 1$, the DE has some solution over some open interval that contains x_0 for any $(x_0, y_0) \in (-\infty, \infty) \times [1, \infty)$. (b) Since $f_y = \partial f / \partial y$ is continuous in y for $y > 1$, the DE has a unique solution over some open interval that contains x_0 for any $(x_0, y_0) \in (-\infty, \infty) \times (1, \infty)$.

Ex. 18. $y' = f(x, y) = 3x(y-1)^{1/3}$, $y(0) = 1$. Compute

$$f_y = \frac{x}{(y-1)^{2/3}}$$

discontinuous at $y = 1$ indicating the possibility of non-unique solutions to the IVP. Note that $y_1 = 1$ is a first solution. The DE is separable, leading to

$$\int \frac{\mathrm{d}y}{(y-1)^{1/3}} = \int 3x\,\mathrm{d}x + \frac{3}{2}c \Rightarrow \frac{3}{2}(y-1)^{2/3} = \frac{3}{2}x^2 + \frac{3}{2}c.$$

The initial condition $y(0) = 1$ implies $c = 0$, hence solutions of $(y-1)^{2/3} = x^2$ are solutions of the IVP. Two such solutions are $y_{2,3} = 1 \pm x^3$. This gives three solutions. Now construct an additional six by choosing different branches, i.e.,

$$y_4 = \begin{cases} 1 & x \leqslant 0 \\ 1 + x^3 & x > 0 \end{cases}, \quad y_5 = \begin{cases} 1 + x^3 & x < 0 \\ 1 & x \geqslant 0 \end{cases},$$

$$y_6 = \begin{cases} 1 & x \leqslant 0 \\ 1 - x^3 & x > 0 \end{cases}, \quad y_7 = \begin{cases} 1 - x^3 & x < 0 \\ 1 & x \geqslant 0 \end{cases},$$

$$y_8 = \begin{cases} 1 - x^3 & x \leqslant 0 \\ 1 + x^3 & x > 0 \end{cases}, \quad y_9 = \begin{cases} 1 + x^3 & x < 0 \\ 1 - x^3 & x \geqslant 0 \end{cases}.$$

4. Exercises 1-4, p. 79

Ex. 1. $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 6x^2y^2\,\mathrm{d}x + 4x^3y\,\mathrm{d}y$. Verify exactness condition $M_y = N_x$

$$M_y = 12x^2y,\ N_x = 12x^2y,\ M_y = N_x \Rightarrow \text{exact differential.}$$

Ex. 2. $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = (3y\cos x + 4x\,e^x + 2x^2\,e^x)\,\mathrm{d}x + (3\sin x + 3)\,\mathrm{d}y$.

$$M_y = 3\cos x,\ N_x = 3\cos x,\ M_y = N_x \Rightarrow \text{exact differential.}$$

Ex. 3. $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = 14x^2y^3\,\mathrm{d}x + 21x^2y^2\,\mathrm{d}y$.

$$M_y = 42x^2y^2,\ N_x = 42x\,y^2,\ M_y \neq N_x \Rightarrow \text{not an exact differential.}$$

Ex. 4. $M(x, y)\mathrm{d}x + N(x, y)\mathrm{d}y = (2x - 2y^2)\,\mathrm{d}x + (12\,y^2 - 4x\,y)\,\mathrm{d}y$.

$$M_y = -4y,\ N_x = -4y,\ M_y = N_x \Rightarrow \text{exact differential.}$$