Homework 4

Due date: Feb 6, 2020, 11:55PM.

Bibliography: Trench Chap. 4. The first exercise in each problem set is solved for you to use as a model.

1. Exercises 1-5, p. 138

Ex 1. Half life $\tau = 3200$ years, initial amount $Q_0 = 20$ g. Radioactive decay model $Q' = -kQ$, $k = (\ln 2)/\tau$ has solution

$$Q(t) = e^{-kt} Q_0 = \exp\left[-\frac{t}{\tau}\ln 2\right] Q_0 = 20 \times 2^{-t/\tau} \text{ g}.$$

Ex 2. $\tau = 2$ days

$$Q(t)/Q_0 = e^{-kt} = 0.1 \Rightarrow t = -\frac{1}{k}\ln 0.1 = -\frac{\tau}{\ln 2}\ln 0.1 = -\frac{2}{\ln 2}\ln 0.1 = 6.64 \text{ days}$$

Ex 3. State condition

$$Q(10)/Q_0 = e^{-10k} = 0.75 \Rightarrow k = \frac{\ln 2}{\tau} = -\frac{1}{10}\ln 0.75 \Rightarrow \tau = -\frac{10\ln 2}{\ln 0.75} = 24.1 \text{ minutes}$$

Ex 4. State conditions

$$\frac{100\, Q(0)}{Q(0) + P} = p_0, \frac{100\, Q(t)}{Q(t) + P} = p_1,$$

with $Q(0)$ initial amount of radioactive substance, $Q(t)$ final amount, and P non-radioactive substance, assumed constant in time. Eliminate P, and replace $Q(t) = e^{-kt} Q(0)$

Ex 5.

2. Exercises 15-19, pp. 138-9

Ex 15. Gold creation rate $r = 1$ oz/hr, theft rate $s = \frac{1}{20} W(t)$ oz/hr. Model equation $W' = r - s = r - \frac{1}{20} W$, and solution to homogeneous problem $W' + \frac{1}{20} W = 0$ is $W_h = e^{-t/20}$. Variation of parameters, $W = u W_h$

$$u' W_h = r \Rightarrow u' = r e^{t/20} \Rightarrow u = 20 r e^{t/20} + c \Rightarrow W(t) = 20r + c e^{-t/20}.$$

Initial condition $W(0) = 20r + c = 1 \Rightarrow c = 1 - 20r \Rightarrow W(t) = 20r(1 - e^{-t/20}) + e^{-t/20}$, $\lim_{t\to\infty} W(t) = 20r$.

Check in Maxima

```
(%i3) ode2('diff(W,t)=r-W/20,W,t);
```

(%o3) $W = e^{-\frac{t}{20}} \left(20\, r\, e^{\frac{t}{20}} + \%c\right)$

```
(%i4) ic1(%,t=0,W=1);
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(%o4) $W = e^{-\frac{t}{20}} \left(20\, r\, e^{\frac{t}{20}} - 20\, r + 1\right)$

```
(%i5)
```

Ex 16.

Ex 17.

Ex 18.

Ex 19.

3. Exercises 1-5, p.148

Ex 1. Room temperature $T_0 = 70°$ F, freezer temperature $T_1 = 12°$ F. Model equation for thermometer temperature $T(t)$ is $T' = -k(T - T_1), T(0) = T_0$. Solution is $T = T_1 + (T_0 - T_1)e^{-kt}$. From $T(1/2) = 40$ obtain

$$12 + 58e^{-k/2} = 40 \Rightarrow k = -2\ln\frac{28}{58} = \ln\left(\frac{29}{14}\right)^2.$$

Evaluate

$$T(2) = 12 + 58\exp\left[-2\ln\left(\frac{29}{14}\right)^2\right] = 12 + 58\left(\frac{14}{29}\right)^4 = 15.2°\,\text{F}.$$

Note that the displayed measurement exhibits a measurable number of digits after the decimal point, rather than reproducing the full, but physically impossible to measure, numerical approximation from the calculation below.

```
(%i13) float(12+58*(14/29)^4)
(%o13) 15.15027266390586
(%i14)
```

Ex 2.

Ex 3.

Ex 4.

Ex 5.

4. Exercises 1-5, p. 160

Ex 1. Weight $m = 192\,\text{lb} = 87.3\text{kg}$, model equation is $mv' = mg - kv$, with $g = 9.8m/s^2$, $k = 2.5\,\text{lbf}\,s/\text{ft} = 36.54\,\text{kg}/s$, with solution $v(t) = mg/k\,[\exp(-kt/m) - 1]$, and terminal velocity $\lim_{t\to\infty} v(t) = mg/k = 23.4m/s$.

Ex 2.

Ex 3.

Ex 4.

Ex 5.