Homework 5 Solution

Due date: Feb 13, 2020, 11:55PM.

Bibliography: Lesson07.pdf Lesson08.pdf. The first exercise in each problem set is solved for you to use as a model.

1. Consider $\mathcal{V}=\mathbb{R}^3$, $\mathcal{S}=\mathbb{R}$. Establish whether for given operations $\oplus,\odot$ $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is a vector space or not.

Ex 1. $\boldsymbol{x}=(x_1,x_2,x_3)$, $\boldsymbol{y}=(y_1,y_2,y_3)$, $\boldsymbol{x},\boldsymbol{y}\in\mathcal{V}=\mathbb{R}^3$, $\boldsymbol{x}\oplus\boldsymbol{y}=(2x_1+2y_1,2x_2+2y_2,2x_3+2y_3)$, and with $\alpha\in\mathcal{S}=\mathbb{R}$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$

Solution. Check associativity, $(\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}=\boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})$. Compute

$$\boldsymbol{u}=\boldsymbol{x}\oplus\boldsymbol{y}=(2x_1+2y_1,2x_2+2y_2,2x_3+2y_3)\quad \boldsymbol{v}=\boldsymbol{y}\oplus\boldsymbol{z}=(2y_1+2z_1,2y_2+2z_2,2y_3+2z_3)$$

$$\boldsymbol{u}+\boldsymbol{z}=(4x_1+4y_1+2z_1,4x_2+4y_2+2z_2,4x_3+4y_3+2z_2)$$

$$\boldsymbol{x}+\boldsymbol{v}=(2x_1+4y_1+4z_1,2x_2+4y_2+4z_2,2x_2+4y_2+4z_2)$$

Since $\boldsymbol{u}+\boldsymbol{z}\neq\boldsymbol{x}+\boldsymbol{v}$, associativity is not satisified and $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is not a vector space. Note: it is sufficient to find one unsatisfied property to prove that $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is not a vector space. However to prove that $(\mathcal{V},\mathcal{S},\oplus,\odot)$ is indeed a vector space, all properties must be verified/

Ex 2. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1-y_1,x_2-y_2,x_3-y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Solution. Check associativity

$$\left.\begin{array}{l}\boldsymbol{x}\oplus\boldsymbol{y}=(x_1-y_1,x_2-y_2,x_3-y_3)\\ \boldsymbol{y}\oplus\boldsymbol{z}=(y_1-z_1,y_2-z_2,y_3-z_3)\\ \boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})=(x_1-y_1+z_1,x_2-y_2+z_2,x_3-y_3+z_3)\\ (\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}=(x_1-y_1-z_1,x_2-y_2-z_3,x_3-y_3-z_3)\end{array}\right.\Rightarrow\boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})\neq(\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}$$

Not associative (e.g., $\boldsymbol{x}=\boldsymbol{y}=0$, $\boldsymbol{z}=(1,0,0)$), hence not a vector space.

Ex 3. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Solution. Check closure. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)\in\mathbb{R}^3\checkmark$.

Check existence of null element. Suppose the null element is denoted by $\boldsymbol{n}$. From $\boldsymbol{n}\oplus\boldsymbol{x}=\boldsymbol{x}$ deduce $(n_1+x_1-1,n_2+x_2-1,n_3+x_3-1)=(x_1,x_2,x_3)$ deduce $\boldsymbol{n}=(1,1,1)$. Verify if $\boldsymbol{x}\oplus\boldsymbol{n}=\boldsymbol{n}\oplus\boldsymbol{x}=\boldsymbol{x}$

$$\boldsymbol{x}\oplus\boldsymbol{n}=(x_1,x_2,x_3)\checkmark,\boldsymbol{n}\oplus\boldsymbol{x}=(x_1,x_2,x_3)\checkmark.$$

Check commutativity.

$$\left.\begin{array}{l}\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)\\ \boldsymbol{y}\oplus\boldsymbol{x}=(y_1+x_1-1,y_2+x_2-1,y_3+x_3-1)=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)\end{array}\right.\checkmark$$

Check associativity.

$$\left.\begin{array}{l}\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)\\ \boldsymbol{y}\oplus\boldsymbol{z}=(y_1+z_1-1,y_2+z_2-1,y_3+z_3-1)\\ \boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})=(x_1+y_1+z_1-2,x_2+y_2+z_2-2,x_3+y_3+z_3-2)\\ (\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}=(x_1+y_1+z_1-2,x_2+y_2+z_2-2,x_3+y_3+z_3-2)\end{array}\right.\Rightarrow\boldsymbol{x}\oplus(\boldsymbol{y}\oplus\boldsymbol{z})=(\boldsymbol{x}\oplus\boldsymbol{y})\oplus\boldsymbol{z}\checkmark$$

Check existence of opposite. Let $\boldsymbol{a}$ denote the opposite of $\boldsymbol{x}$. From $\boldsymbol{x}\oplus\boldsymbol{a}=\boldsymbol{n}$ deduce

$$(x_1+a_1-1,x_2+a_2-1,x_2+a_2-1)=(1,1,1)\Rightarrow\boldsymbol{a}=(2-x_1,2-x_2,2-x_3).\checkmark$$

Check distributivity properties.

$$\begin{array}{l}\alpha\odot(\boldsymbol{x}\oplus\boldsymbol{y})=\alpha(x_1+y_1-1,x_2+y_2-1,x_3+y_3-1)=(\alpha x_1+\alpha y_1-\alpha,\alpha x_2+\alpha y_2-\alpha,\alpha x_3+\alpha y_3-\alpha)\\ \alpha\odot\boldsymbol{x}\oplus\alpha\odot\boldsymbol{y}=(\alpha x_1,\alpha x_2,\alpha x_3)\oplus(\alpha y_1,\alpha y_2,\alpha y_3)=(\alpha x_1+\alpha y_1-1,\alpha x_2+\alpha y_2-1,\alpha x_3+\alpha y_3-1)\end{array}$$

Distributivity is not satisfied (e.g., for $\alpha=0$), hence not a vector space.

Note: this exercise shows that the vector space properties have to be carefully checked individually, using the formal definitions without relying on intuition. For example, it is shown that the null element does not need to be $(0,0,0)$. All the commutative group properties were satisfied, but scalar multiplication did not verify one of the distributivity properties. Remember: *don't assume, prove!*.

Ex 4. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1,x_2-y_2,x_3+y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha x_1,\alpha x_2,\alpha x_3)$.

Solution. Check commutativity

$$\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1,x_2-y_2,x_3+y_3)\neq(y_1+x_1,y_2-x_2,y_3+x_3)=\boldsymbol{y}\oplus\boldsymbol{x}\ ,$$

not verified. For example $\boldsymbol{x}=(0,1,0)$, $\boldsymbol{y}=(0,-1,0)$

$$\boldsymbol{x}\oplus\boldsymbol{y}=(0,2,0)\neq(0,-2,0)=\boldsymbol{y}\oplus\boldsymbol{x}\ ,$$

hence not a vector space.

Ex 5. $\boldsymbol{x}\oplus\boldsymbol{y}=(x_1+y_1,x_2+y_2,x_3+y_3)$, $\alpha\odot\boldsymbol{x}=(\alpha+x_1,\alpha x_2,\alpha x_3)$.

Solution. Check distributivity

$$\left.\begin{array}{l}\alpha\odot(\beta\odot\boldsymbol{x})=\alpha\odot(\beta+x_1,\beta x_2,\beta x_3)=(\alpha+\beta+x_1,\alpha\beta x_2,\alpha\beta x_3)\\(\alpha\cdot\beta)\boldsymbol{x}=(\alpha\beta+x_1,\alpha\beta x_2,\alpha\beta x_3)\end{array}\right.\Rightarrow\alpha\odot(\beta\odot\boldsymbol{x})\neq(\alpha\cdot\beta)\boldsymbol{x}$$

not distributive, example $\alpha=\beta=1$, hence not a vector space.

2. Consider $\mathcal{V}\subset\mathbb{R}^{2\times2}$, a subset of all 2 by 2 real-component matrices with operations

$$\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^{2\times2},\boldsymbol{A}=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix},\boldsymbol{B}=\begin{pmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{pmatrix},\boldsymbol{A}\oplus\boldsymbol{B}=\begin{pmatrix}a_{11}+b_{11}&a_{12}+b_{12}\\a_{21}+b_{21}&a_{22}+b_{22}\end{pmatrix}$$

$$\alpha\odot\boldsymbol{A}=\begin{pmatrix}\alpha a_{11}&\alpha a_{12}\\\alpha a_{21}&\alpha a_{22}\end{pmatrix}.$$

Determine whether the following are vector spaces

Ex 1. $\mathcal{V}$ is the set of skew-symmetric matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ \boldsymbol{A}^T=-\boldsymbol{A}$.

Solution. From $\boldsymbol{A}=-\boldsymbol{A}^T$ deduce that

$$\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}=-\begin{pmatrix}a_{11}&a_{21}\\a_{12}&a_{22}\end{pmatrix}\Rightarrow\boldsymbol{A}=\begin{pmatrix}0&a\\-a&0\end{pmatrix}.$$

Verify vector space properties for $\forall\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\in\mathcal{V}$, $\forall\alpha,\beta\in\mathbb{R}$:

Closure.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix}0&a\\-a&0\end{pmatrix}+\begin{pmatrix}0&b\\-b&0\end{pmatrix}=\begin{pmatrix}0&a+b\\-(a+b)&0\end{pmatrix}\in\mathcal{V}\checkmark$$

Associativity.

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\begin{pmatrix}0&a+b\\-(a+b)&0\end{pmatrix}+\begin{pmatrix}0&c\\-c&0\end{pmatrix}=\begin{pmatrix}0&a+b+c\\-(a+b+c)&0\end{pmatrix}$$

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix}0&a\\-a&0\end{pmatrix}+\begin{pmatrix}0&b+c\\-(b+c)&0\end{pmatrix}=\begin{pmatrix}0&a+b+c\\-(a+b+c)&0\end{pmatrix}.\checkmark$$

Identity.

$$\boldsymbol{A}+\boldsymbol{0}=\begin{pmatrix}0&a\\-a&0\end{pmatrix}+\begin{pmatrix}0&0\\0&0\end{pmatrix}=\begin{pmatrix}0&a\\-a&0\end{pmatrix}\checkmark$$

Inverse.

$$\boldsymbol{A}+(-\boldsymbol{A})=\begin{pmatrix}0&a\\-a&0\end{pmatrix}+\begin{pmatrix}0&-a\\a&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}\checkmark$$

Commutativity.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} 0 & a+b \\ -(a+b) & 0 \end{pmatrix}=\begin{pmatrix} 0 & b+a \\ -(b+a) & 0 \end{pmatrix}=\boldsymbol{B}+\boldsymbol{A}.\checkmark$$

Distributivity.

$$\alpha(\boldsymbol{A}+\boldsymbol{B})=\begin{pmatrix} 0 & \alpha(a+b) \\ -\alpha(a+b) & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha a+\alpha b) \\ -\alpha a-\alpha b) & 0 \end{pmatrix}=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}\checkmark$$

$$(\alpha+\beta)\boldsymbol{A}=\begin{pmatrix} 0 & (\alpha+\beta)a \\ -(\alpha+\beta)a & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha a+\beta a \\ -\alpha a-\beta a & 0 \end{pmatrix}=\alpha\boldsymbol{A}+\beta\boldsymbol{A}\checkmark$$

$$\alpha(\beta\boldsymbol{A})=\alpha\begin{pmatrix} 0 & \beta a \\ -\beta a & 0 \end{pmatrix}=\begin{pmatrix} 0 & \alpha\beta a \\ -\alpha\beta a & 0 \end{pmatrix}=(\alpha\beta)\boldsymbol{A}.\checkmark$$

All properties are verified, hence skew-symmetric matrices form a vector space.

Ex 2. $\mathcal{V}$ is the set of upper-triangular matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ a_{21}=0$.

Solution. (Tip: when drafting answers such as this, it is convenient to cope and paste the above template and also copy and paste various intermediate results, but do be careful and ensure that the specific definitions in the new exercise are adhered to.)

Verify vector space properties for $\forall\boldsymbol{A},\boldsymbol{B},\boldsymbol{C}\in\mathcal{V}$, $\forall\alpha,\beta\in\mathbb{R}$:

Closure.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{pmatrix}\in\mathcal{V}\checkmark$$

Associativity.

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{pmatrix}+\begin{pmatrix} c_{11} & c_{12} \\ 0 & c_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ 0 & a_{22}+b_{22}+c_{22} \end{pmatrix}$$

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ 0 & b_{22}+c_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ 0 & a_{22}+b_{22}+c_{22} \end{pmatrix}.\checkmark$$

Identity.

$$\boldsymbol{A}+\boldsymbol{0}=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\checkmark$$

Inverse.

$$\boldsymbol{A}+(-\boldsymbol{A})=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}+\begin{pmatrix} -a_{11} & -a_{12} \\ 0 & -a_{22} \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\checkmark$$

Commutativity.

$$\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ 0 & a_{22}+b_{22} \end{pmatrix}=\begin{pmatrix} b_{11}+a_{11} & b_{12}+a_{12} \\ 0 & b_{22}+a_{22} \end{pmatrix}=\boldsymbol{B}+\boldsymbol{A}.\checkmark$$

Distributivity.

$$\alpha(\boldsymbol{A}+\boldsymbol{B})=\begin{pmatrix} \alpha(a_{11}+b_{11}) & \alpha(a_{12}+b_{12}) \\ 0 & \alpha(a_{22}+b_{22}) \end{pmatrix}=\begin{pmatrix} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} \\ 0 & \alpha a_{22}+\alpha b_{22} \end{pmatrix}=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}\checkmark$$

$$(\alpha+\beta)\boldsymbol{A}=\begin{pmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} \\ 0 & (\alpha+\beta)a_{22} \end{pmatrix}=\begin{matrix} \alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12} \\ 0 & \alpha a_{22}+\beta b_{22} \end{matrix}=\alpha\boldsymbol{A}+\beta\boldsymbol{A}\checkmark$$

$$\alpha(\beta\boldsymbol{A})=\alpha\begin{pmatrix} \beta a_{11} & \beta a_{12} \\ 0 & \beta a_{22} \end{pmatrix}=\begin{pmatrix} \alpha\beta a_{11} & \alpha\beta a_{12} \\ 0 & \alpha\beta a_{22} \end{pmatrix}=(\alpha\beta)\boldsymbol{A}.\checkmark$$

All properties are verified, hence upper-triangular matrices form a vector space.

Ex 3. $\mathcal{V}$ is the set of symmetric matrices, $\boldsymbol{A}\in\mathcal{V}\Rightarrow\ \boldsymbol{A}^T=\boldsymbol{A}$.

Solution. From $\boldsymbol{A} = \boldsymbol{A}^T$ deduce that

$$\left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array}\right) = \left(\begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \end{array}\right) \Rightarrow \boldsymbol{A} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right).$$

Verify vector space properties for $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{V}$, $\forall \alpha, \beta \in \mathbb{R}$:

Closure.

$$\boldsymbol{A} + \boldsymbol{B} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right) + \left(\begin{array}{cc} b_{11} & b_{12} \\ b_{12} & b_{22} \end{array}\right) = \left(\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{array}\right) \in \mathcal{V} \checkmark$$

Associativity.

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C} = \left(\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{array}\right) + \left(\begin{array}{cc} c_{11} & c_{12} \\ c_{12} & c_{22} \end{array}\right) = \left(\begin{array}{cc} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{12}+b_{12}+c_{12} & a_{22}+b_{22}+c_{22} \end{array}\right)$$

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C}) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right) + \left(\begin{array}{cc} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{12}+c_{12} & b_{22}+c_{22} \end{array}\right) = \left(\begin{array}{cc} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{12}+b_{12}+c_{12} & a_{22}+b_{22}+c_{22} \end{array}\right). \checkmark$$

Identity.

$$\boldsymbol{A} + \boldsymbol{0} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right) + \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right) \checkmark$$

Inverse.

$$\boldsymbol{A} + (-\boldsymbol{A}) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right) + \left(\begin{array}{cc} -a_{11} & -a_{12} \\ -a_{12} & -a_{22} \end{array}\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right) \checkmark$$

Commutativity.

$$\boldsymbol{A} + \boldsymbol{B} = \left(\begin{array}{cc} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{12}+b_{12} & a_{22}+b_{22} \end{array}\right) = \left(\begin{array}{cc} b_{11}+a_{11} & b_{12}+a_{12} \\ b_{12}+a_{12} & b_{22}+a_{22} \end{array}\right) = \boldsymbol{B} + \boldsymbol{A}. \checkmark$$

Distributivity.

$$\alpha(\boldsymbol{A}+\boldsymbol{B}) = \left(\begin{array}{cc} \alpha(a_{11}+b_{11}) & \alpha(a_{12}+b_{12}) \\ \alpha(a_{12}+b_{12}) & \alpha(a_{22}+b_{22}) \end{array}\right) = \left(\begin{array}{cc} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} \\ \alpha a_{12}+\alpha b_{12} & \alpha a_{22}+\alpha b_{22} \end{array}\right) = \alpha\boldsymbol{A} + \alpha\boldsymbol{B} \checkmark$$

$$(\alpha+\beta)\boldsymbol{A} = \left(\begin{array}{cc} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} \\ (\alpha+\beta)a_{12} & (\alpha+\beta)a_{22} \end{array}\right) = \begin{array}{cc} \alpha a_{11}+\beta b_{11} & \alpha a_{12}+\beta b_{12} \\ \alpha a_{12}+\beta b_{12} & \alpha a_{22}+\beta b_{22} \end{array} = \alpha\boldsymbol{A} + \beta\boldsymbol{A} \checkmark$$

$$\alpha(\beta\boldsymbol{A}) = \alpha\left(\begin{array}{cc} \beta a_{11} & \beta a_{12} \\ \beta a_{12} & \beta a_{22} \end{array}\right) = \left(\begin{array}{cc} \alpha\beta a_{11} & \alpha\beta a_{12} \\ \alpha\beta a_{12} & \alpha\beta a_{22} \end{array}\right) = (\alpha\beta)\boldsymbol{A}. \checkmark$$

All properties are verified, hence upper-triangular matrices form a vector space.

3. Determine whether the set $\mathcal{S}$ is linearly dependent or independent within the vector space $\mathcal{V}$

Ex 1.

$$\mathcal{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2\} = \left\{\left(\begin{array}{c} 2 \\ 1 \\ 3 \end{array}\right), \left(\begin{array}{c} 0 \\ -1 \\ 1 \end{array}\right)\right\}, \mathcal{V} = \mathbb{R}^3$$

Solution. The first equation of the system $a_1\boldsymbol{u}_1 + a_2\boldsymbol{u}_2 = \boldsymbol{0}$ is $2a_1 + 0a_2 = 0 \Rightarrow a_1 = 0$. The second equation then states $-a_2 = 0$, hence $a_1 = a_2 = 0$, and $\boldsymbol{u}_1, \boldsymbol{u}_2$ are linearly independent.

Ex 2.

$$\mathcal{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \left\{\left(\begin{array}{c} 2 \\ 1 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 8 \\ -3 \end{array}\right)\right\}, \mathcal{V} = \mathbb{R}^2$$

Solution. Since $(-3)\boldsymbol{u}_1 + 14\boldsymbol{u}_2 = \boldsymbol{u}_3$, the vectors are linearly dependent.

Ex 3.

$$\mathcal{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}, \mathcal{V} = \mathbb{R}^3$$

Solution. (The missing $\boldsymbol{u}_3$ was a typo). Since $\boldsymbol{u}_1 + \boldsymbol{u}_2 = \boldsymbol{u}_3$, the vectors are linearly dependent.

4. Determine whether the set $\mathcal{S}$ is linearly dependent or independent within the vector space $\mathcal{V}$. Here $\mathcal{P}_n$ is the set of polynomials of degree at most n.

Ex 1.

$$\mathcal{S} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\} = \{1, 2x^2 + x + 2, -x^2 + x\}, \mathcal{V} = \mathcal{P}_2$$

Solution. Denote $\boldsymbol{q} = a_1\boldsymbol{p}_1 + a_2\boldsymbol{p}_2 + a_3\boldsymbol{p}_3$, and consider the equality $\boldsymbol{q} = \mathbf{0}$. Note that $\mathbf{0}$ is the zero polynomial, i.e. $\boldsymbol{q}(x) = \mathbf{0}(x) \Rightarrow$

$$a_1 + a_2(2x^2 + x + 2) + a_3(-x^2 + x) = 0 \text{ for all } x$$

For $x = 0$ obtain $a_1 + a_2 = 0$. Subsequently for $x = 1$ obtain $a_1 + 5a_2 = 0$. Subtract to obtain $4a_2 = 0 \Rightarrow a_2 = 0$, and then $a_1 = 0$. Then for $x = -1$ obtain $-2a_3 = 0 \Rightarrow a_3 = 0$. The only choice of a_1, a_2, a_3 to have $a_1\boldsymbol{p}_1 + a_2\boldsymbol{p}_2 + a_3\boldsymbol{p}_3 = \mathbf{0}$ is $a_1 = a_2 = a_3 = 0$, hence $\mathcal{S}$ is a linearly independent set of vectors.

Ex 2.

$$\mathcal{S} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3\} = \{2, x, x^3 + 2x^2 - 1\}, \mathcal{V} = \mathcal{P}_3$$

Solution. Denote $\boldsymbol{q} = a_1\boldsymbol{p}_1 + a_2\boldsymbol{p}_2 + a_3\boldsymbol{p}_3$, and consider the equality $\boldsymbol{q} = \mathbf{0}$,

$$2a_1 - a_3 + a_2x + 2a_3x^2 + a_3x^3 = 0.$$

Evaluate at $x = -1, 0, 1$ to obtain a system

$$\left\{ \begin{array}{lll} 2a_1 - a_3 - a_2 + 2a_3 - a_3 & = & 0 \\ 2a_1 - a_3 & = & 0 \\ 2a_1 - a_3 + a_2 + 2a_3 + a_3 & = & 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{lll} 2a_1 - a_2 & = & 0 \\ 2a_1 - a_3 & = & 0 \\ 2a_1 + a_2 + 2a_3 & = & 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{lll} 2a_1 - a_2 & = & 0 \\ a_2 - a_3 & = & 0 \\ 2a_2 + 2a_3 & = & 0 \end{array} \right. \Rightarrow \begin{array}{lll} 2a_1 - a_2 & = & 0 \\ a_2 - a_3 & = & 0 \\ 4a_3 & = & 0 \end{array}$$

with $a_1 = a_2 = a_3$ the solution, hence $\mathcal{S}$ is a linearly independent set.

Ex 3.

$$\mathcal{S} = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3, \boldsymbol{p}_4\} = \{x, x^2, x^2 + 2x, x^3 - x + 1\}, \mathcal{V} = \mathcal{P}_3$$

Solution. Since $\boldsymbol{p}_3 = \boldsymbol{p}_2 + 2\boldsymbol{p}_1$, $\mathcal{S}$ is a linearly dependent set.