Test 1 Solution

Solve the following problems (3 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. Organize your computation and writing so the solution you present is readily legible. No credit is awarded for statement of the final answer to a problem without presentation of solution procedure.

Study the following test solution to familiarize yourself with the formulation of succint, correct answers to mathematical problems. The presentation of each solution is repeated with additional comments highlighted in dark red. These comments present the motivation underlying answer formulation. Remember: your main goal is demonstrate understanding of mathematical concepts, not to just present a final answer. Always ask yourself: "how do I show knowledge of this topic?", not "what is the answer to this problem?".

1. Verify for $c \in \mathbb{R}$ $y(x) = c^2 + cx + 2c + 1$ satisfies

$$y' = \frac{-x - 2 + \sqrt{x^2 + 4x + 4y}}{2} \qquad (1)$$

on some open interval. Identify the open interval. Verify that $y_1 = -x(x+4)/4$ also satisfies (1) on some open interval. Identify the interval.

Solution. A DE $y' = f(x, y)$ has a solution over some open rectangle $R = (a, b) \times (c, d)$ if $f(x, y)$ is continuous in y over R. If $f_y(x, y) = \partial f / \partial y$ is also continuous in R the DE has a unique solution. Compute

$$f_y = \frac{\partial}{\partial y}\left(\frac{-x - 2 + \sqrt{x^2 + 4x + 4y}}{2}\right) = \frac{1}{2}\frac{\partial}{\partial y}\left(\sqrt{x^2 + 4x + 4y}\right) = \frac{1}{\sqrt{x^2 + 4x + 4y}},$$

and note that f_y is not continuous when $x^2 + 4x + 4y = 0$.

Verify that $y(x) = c^2 + cx + 2c + 1$ is a solution. Compute:

$$y' = c, \;\; x^2 + 4x + 4y = x^2 + 4x + 4(c^2 + cx + 2c + 1) = x^2 + 4(x + c) + [2(c+1)]^2 = [x + 2(c+1)]^2$$

$$y' = c = \frac{-x - 2 + \sqrt{[x + 2(c+1)]^2}}{2} = \frac{-x - 2 + x + 2(c+1)}{2} = c. \checkmark$$

To obtain the linear function $y(x) = cx + (c+1)^2$ as a solution, the initial condition could be chosen as $x = 0$, $y = (c+1)^2$, and $R = (0, 1) \times ((c+1)^2, \infty)$.

Verify that $y_1 = -x(x+4)/4$ is a solution. Compute:

$$y_1' = -\frac{1}{2}x - 1, x^2 + 4x + 4y_1 = x^2 + 4x - x(x+4) = 0,$$

$$y' = -\frac{1}{2}x - 1 = \frac{-x - 2 + \sqrt{0}}{2} = -\frac{1}{2}x - 1. \checkmark$$

To obtain y_1 as a solution, choose initial condition $y(0) = 0$, and $R = (0, 1) \times (0, -\infty)$.

Commented solution. A DE $y' = f(x, y)$ has a solution over some open rectangle $R = (a, b) \times (c, d)$ if $f(x, y)$ is continuous in y over R [*show knowledge of the solution existence theorem*]. If $f_y(x, y) = \partial f / \partial y$ is also continuous in R the DE has a unique solution [*show knowledge of the solution uniqueness theorem, relevant here because you are asked to show that two solutions verify the DE*]. Compute

$$f_y = \frac{\partial}{\partial y}\left(\frac{-x - 2 + \sqrt{x^2 + 4x + 4y}}{2}\right) = \frac{1}{2}\frac{\partial}{\partial y}\left(\sqrt{x^2 + 4x + 4y}\right) = \frac{1}{\sqrt{x^2 + 4x + 4y}},$$

and note that f_y is not continuous when $x^2+4x+4y=0$. [*Carry out the computations suggested by the existence and uniqueness theorems, identify possibility of non-unique solution*].

Verify that $y(x)=c^2+cx+2c+1$ is a solution. Compute [*carry out calculus computations*]:

$$y'=c,\ \ x^2+4x+4y=x^2+4x+4(c^2+cx+2c+1)=x^2+4(x+c)+[2(c+1)]^2=[x+2(c+1)]^2$$

$$y'=c=\frac{-x-2+\sqrt{[x+2(c+1)]^2}}{2}=\frac{-x-2+x+2(c+1)}{2}=c.\checkmark$$

To obtain the linear function $y(x)=cx+(c+1)^2$ as a solution, the initial condition could be chosen as $x=0$, $y=(c+1)^2$, and $R=(-1,1)\times((c+1)^2,\infty)$. [*Choose some arbitrary domain that selects the monotone increasing branch*]

Verify that $y_1=-x(x+4)/4$ is a solution. Compute[*carry out calculus computations*]:

$$y_1'=-\frac{1}{2}x-1, x^2+4x+4y_1=x^2+4x-x(x+4)=0,$$
$$y'=-\frac{1}{2}x-1=\frac{-x-2+\sqrt{0}}{2}=-\frac{1}{2}x-1.\checkmark$$

To obtain y_1 as a solution, choose initial condition $y(0)=0$, and $R=(-1,1)\times(0,-\infty)$. [*Choose some arbitrary domain that contains the monotone decreasing branch*]

2. Use variation of parameters and separation of variables to solve

$$xy'-2y=\frac{x^6}{y+x^2}.$$

Solution. Solve the homogeneous problem $xy'-2y=0$

$$\frac{y'}{y}=\frac{2}{x}\Rightarrow y_h(x)=x^2.$$

By variation of parameters, seek solution of form $y(x)=u(x)\,y_h(x)=x^2\,u(x)$

$$x(x^2\,u)'-2x^2u=\frac{x^6}{x^2(u+1)}\Rightarrow x^3\,u'=\frac{x^4}{u+1}\Rightarrow (u+1)u'=x\Rightarrow$$
$$\int (u+1)\,\mathrm{d}u=\int x\,\mathrm{d}x+\frac{c}{2}\Rightarrow (u+1)^2=x^2+c\Rightarrow u=-1\pm\sqrt{x^2+c}.$$

Equation has multiple solutions since in $y'=f(x,y)=2y/x+x^5/(y+x^2)$, f is not continuous when $y+x^2=0$.

Commented solution. Solve the homogeneous problem $xy'-2y=0$ [*show knowledge of variation of parameters first step: solve the homogeneous problem*]

$$\frac{y'}{y}=\frac{2}{x}\Rightarrow y_h(x)=x^2.$$

By variation of parameters, seek solution of form $y(x)=u(x)\,y_h(x)=x^2\,u(x)$ [*show knowledge of variation of parameters second step: assume solution is a modification of y_h*]

$$x(x^2\,u)'-2x^2u=\frac{x^6}{x^2(u+1)}\Rightarrow x^3\,u'=\frac{x^4}{u+1}\Rightarrow (u+1)u'=x\Rightarrow$$
$$\int (u+1)\,\mathrm{d}u=\int x\,\mathrm{d}x+\frac{c}{2}\Rightarrow (u+1)^2=x^2+c\Rightarrow u=-1\pm\sqrt{x^2+c}.$$

[*Carry out calculations*]. Equation has multiple solutions since in $y'=f(x,y)=2y/x+x^5/(y+x^2)$, f is not continuous when $y+x^2=0$.[*Show recognition of possible multiple solutions*]

3. Find all (x_0, y_0) for which the initial value problem

$$y' = \frac{x^2 + y^2}{\ln(xy)}, \; y(x_0) = y_0$$

has a solution on some open interval that contains x_0.

Solution. A solution exists for an interval $x \in (a, b)$, over $f(x, y) = (x^2 + y^2)/\ln(xy)$ would be continuous in y. The function f is discontinuous at points (x_0, y_0) when $\ln(x_0 y_0) = 0 \Rightarrow y_0 = 1/x_0$, or when $x_0 = 0$ or when $y_0 = 0$, and is undefined for $x_0 y_0 < 0$. A solution will exist in some interval $(0, a)$ with $a > x_0$ for $x_0 > 0$, and also in some interval $(b, 0)$ with $b < x_0$ for $y_0 < 0, y_0 \neq 1/x_0, x_0 < 0$.

Commented solution. A solution exists for an interval $x \in (a, b)$, over $f(x, y) = (x^2 + y^2)/\ln(xy)$ would be continuous in y [*state theorem for solution existence*]. The function f is discontinuous at points (x_0, y_0) when $\ln(x_0 y_0) = 0 \Rightarrow y_0 = 1/x_0$, or when $x_0 = 0$ or when $y_0 = 0$, and is undefined for $x_0 y_0 < 0$ [*identify points where existence theorem conditions are not met*]. A solution will exist in some interval $(0, a)$ with $a > x_0$ for $x_0 > 0$, and also in some interval $(b, 0)$ with $b < x_0$ for $y_0 < 0, y_0 \neq 1/x_0, x_0 < 0$ [*find intervals that avoid discontinuous f*].

4. Find all functions N such that $(\ln(xy) + 2y \sin x)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ is exact.

Solution. The differential $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}x$ is exact if $M_y = N_x$, hence

$$N_x = M_y = \frac{1}{y} + 2\sin x \Rightarrow N = \frac{x}{y} - 2\cos x + f(y),$$

with f some arbitrary function of y.

Commented solution. The differential $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}x$ is exact if $M_y = N_x$ [*state condition for an exact differential*], hence

$$N_x = M_y = \frac{1}{y} + 2\sin x \Rightarrow N = \frac{x}{y} - 2\cos x + f(y),$$

with f some arbitrary function of y [*carry out calculation*].