Homework 10

Due date: April 2, 2020, 11:55PM.

Bibliography: Trench, 10.1-6.4

1. Exercise 5a-d, p. 516

a) $x'''=f(t,x,y,y'),\ y''=g(t,y,y')$. Introduce notations

$$z_1=x,\, z_2=x',\, z_3=x'',\, z_4=y,\, z_5=y'$$

and obtain system of 5 first-order equations

$$\begin{cases} z_1'=z_2 \\ z_2'=z_3 \\ z_3'=f(t,z_1,z_4,z_5) \\ z_4'=z_5 \\ z_5'=g(t,z_4,z_5) \end{cases}.$$

b) $u'=f(t,u,v,v',w'), v''=g(t,u,v,v',w), w''=h(t,u,v,v',w,w')$. Introduce notations

$$z_1=u,\, z_2=v,\, z_3=v',\, z_4=w,\, z_5=w'$$

and obtain system of 5 first-order equations

$$\begin{cases} z_1'=f(t,z_1,z_2,z_5) \\ z_2'=z_3 \\ z_3'=g(t,z_1,z_2,z_3,z_4) \\ z_4'=z_5 \\ z_5'=h(t,z_1,z_2,z_3,z_4,z_5) \end{cases}.$$

c) $y'''=f(t,y,y',y'')$. Introduce notations

$$z_1=y,\, z_2=y',\, z_3=y''$$

and obtain system of 3 first-order equations

$$\begin{cases} z_1'=z_2 \\ z_2'=z_3 \\ z_3'=f(t,z_1,z_2) \end{cases}.$$

d) $y^{(4)}=f(t,y)$. Introduce notations

$$z_1=y,\, z_2=y',\, z_3=y'',\, z_4=y'''$$

and obtain system of 4 first-order equations

$$\begin{cases} z_1'=z_2 \\ z_2'=z_3 \\ z_3'=z_4 \\ z_4'=f(t,z_1) \end{cases}.$$

2. Exercises 8a,b,e,f, p. 521

a) From

$$\boldsymbol{Y}=\begin{pmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{pmatrix}, \boldsymbol{A}=\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

compute

$$\boldsymbol{Y}' = \begin{pmatrix} 6e^{6t} & -2e^{-2t} \\ 6e^{6t} & 2e^{-2t} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{pmatrix} = \begin{pmatrix} (2+4)e^{6t} & (2-4)e^{-2t} \\ (4+2)e^{6t} & (4-2)e^{-2t} \end{pmatrix} \checkmark.$$

b) From

$$\boldsymbol{Y} = \begin{pmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}$$

compute

$$\boldsymbol{Y}' = \begin{pmatrix} -4e^{-4t} & -6e^{3t} \\ -4e^{-4t} & 15e^{3t} \end{pmatrix} = \begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & -2e^{3t} \\ e^{-4t} & 5e^{3t} \end{pmatrix} = \begin{pmatrix} (-2-2)e^{-4t} & (4-10)e^{3t} \\ (-5+1)e^{-4t} & (10+5)e^{3t} \end{pmatrix} \checkmark.$$

e) From

$$\boldsymbol{Y} = \begin{pmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}$$

compute $\boldsymbol{Y}'$

$$\begin{pmatrix} e^{t} & -e^{-t} & -2e^{-2t} \\ e^{t} & 0 & 4e^{-2t} \\ 0 & 0 & -2e^{-2t} \end{pmatrix} = \begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} e^{t} & e^{-t} & e^{-2t} \\ e^{t} & 0 & -2e^{-2t} \\ 0 & 0 & e^{-2t} \end{pmatrix} = \begin{pmatrix} (-1+2)e^{t} & (-1)e^{-t} & (-1-4+3)e^{-2t} \\ (1)e^{t} & 0 & (-2+6)e^{-2t} \\ 0 & 0 & (-2)e^{-2t} \end{pmatrix} \checkmark.$$

f) (same procedure as above)

3. Exercises 7,8, p. 528-9

   Ex 7) (a) is verified in (2a) above.

   (b) In the general solution

$$\boldsymbol{y}(t) = c_1 \begin{pmatrix} e^{6t} \\ e^{6t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$$

impose initial condition

$$\boldsymbol{y}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 9 \end{pmatrix}$$

a linear system for (c_1, c_2), with solution $c_1 = 3$, $c_2 = -6$, hence

   c) Replace $t = 0$ in

$$\boldsymbol{Y}(t) = \begin{pmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{pmatrix}$$

to obtain

$$\boldsymbol{Y}(0) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

with inverse

$$[\boldsymbol{Y}(0)]^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and compute

$$\boldsymbol{Y}(t)[\boldsymbol{Y}(0)]^{-1}\boldsymbol{k} = \begin{pmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ 9 \end{pmatrix} = \begin{pmatrix} e^{6t} & e^{-2t} \\ e^{6t} & -e^{-2t} \end{pmatrix} \begin{pmatrix} 3 \\ -6 \end{pmatrix} \checkmark$$

4. Exercises 16,17,20,21, p. 541

16.

$$\boldsymbol{A}=\begin{pmatrix} -7 & 4 \\ -6 & 7 \end{pmatrix}, p(\lambda)=\det|\lambda I-\boldsymbol{A}|=\begin{vmatrix} \lambda+7 & -4 \\ 6 & \lambda-7 \end{vmatrix}=\lambda^2-49+24=\lambda^2-25\Rightarrow\lambda_{1,2}=\pm5$$

$$e^{\lambda_1 t}\boldsymbol{x}_1, e^{\lambda_2 t}\boldsymbol{x}_2, \lambda_1=-5$$

$$\boldsymbol{A}-\lambda_1\boldsymbol{I}=\begin{pmatrix} -2 & 4 \\ -6 & 12 \end{pmatrix}\sim\begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}\Rightarrow\begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_{11} \\ x_{21} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}\Rightarrow -2x_{11}+4x_{21}=0\Rightarrow\boldsymbol{x}_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x}=\lambda\boldsymbol{x}$$

$$\boldsymbol{y}(t)=c_1e^{\lambda_1 t}\boldsymbol{x}_1+c_2e^{\lambda_2 t}\boldsymbol{x}_2, \boldsymbol{y}(0)=c_1\boldsymbol{x}_1+c_2\boldsymbol{x}_2=\begin{pmatrix} 2 \\ -4 \end{pmatrix}\Rightarrow$$

$$c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}+c_2\begin{pmatrix} 1/3 \\ 1 \end{pmatrix}=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

$$\boldsymbol{k}=\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$