Homework 11 Solution

Due date: April 9, 2020, 11:55PM.

Bibliography: Lesson 21, Trench, 3.1-3.3

1. Exercises 1-5, p. 106
2. Exercises 1-5, p. 124
3. Exercise 13, p. 108
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Solutions

Here is a template for this numerical methods homework.

1. Euler's method. Ex.1. $y' = f(x,y) = 2x^2 + 3y^2 - 2,\ y(2) = 1,\ h = 0.05.$

```
(%i1) f(x,y):=2*x^2+3*y^2-2$
(%i2) h:0.05$ x0:2$ y0:1$
(%i5) for i:1 thru 3 do
        ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1,
          display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 2.05, 1.45]$

$[i, x1, y1] = [2, 2.1, 2.085625]$

$[i, x1, y1] = [3, 2.149999999999999, 3.07909974609375]$

```
(%i6)
```

Ex. 2 $y' = f(x,y) = y + \sqrt{x^2+y^2},\ y(0) = 1,\ h = 0.1.$

```
(%i1) f(x,y):=y+sqrt(x^2+y^2)$
(%i2) h: 0.1$ x0: 0$ y0: 1$
(%i5) for i:1 thru 3 do
        ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1,
          display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 0.1, 1.2]$

$[i, x1, y1] = [2, 0.2, 1.440415945787923]$

$[i, x1, y1] = [3, 0.3, 1.729880994351281]$

```
(%i6)
```

Ex. 3. $y' = f(x,y) = x^2 + y^2 - 3y(x+1),\ y(0) = 2,\ h = 0.05.$

```
(%i6) f(x,y):=x^2+y^2-3*y*(x+1)$
(%i7) h: 0.05$ x0: 0$ y0: 2$
(%i10) for i:1 thru 3 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 0.05, 1.9]$

$[i, x1, y1] = [2, 0.1, 1.781375]$

$[i, x1, y1] = [3, 0.15, 1.64661296953125]$

```
(%i11)
```

Ex. 4. $y' = f(x,y) = (1+x)/(1-y^2),\ y(2) = 3,\ h = 0.1.$

```
(%i11) f(x,y):=(1+x)/(1-y^2)$
(%i12) h: 0.1$ x0: 2$ y0: 3$
(%i15) for i:1 thru 3 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 2.1, 2.9625]$

$[i, x1, y1] = [2, 2.2, 2.922635827523157]$

$[i, x1, y1] = [3, 2.3, 2.880205639458119]$

```
(%i16)
```

Ex. 5. $y' = f(x,y) = \sin(xy) - x^2 y,\ y(1) = \pi,\ h = 0.2.$

```
(%i11) f(x,y):=sin(x*y)-x^2*y$
(%i20) h: 0.2$ x0: 1$ y0: 3.1415$
(%i23) for i:1 thru 3 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 1.2, 3.096399199852611]$

$[i, x1, y1] = [2, 1.4, 3.04516305809343]$

$[i, x1, y1] = [3, 1.6, 2.987143119411182]$

(%i24)

2. Runge-Kutta method. Ex.1. $y' = f(x, y) = 2x^2 + 3y^2 - 2,\ y(2) = 1,\ h = 0.05.$

```
(%i6) f(x,y):=2*x^2+3*y^2-2$
(%i7) h:0.05$ x0:2$ y0:1$
(%i10) for i:1 thru 3 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 2.05, 1.550598189929713]$

$[i, x1, y1] = [2, 2.1, 2.469649728729797]$

$[i, x1, y1] = [3, 2.149999999999999, 4.530350898626433]$

(%i11)

Ex. 2 $y' = f(x, y) = y + \sqrt{x^2 + y^2},\ y(0) = 1,\ h = 0.1.$

```
(%i24) f(x,y):=y+sqrt(x^2+y^2)$
(%i25) h: 0.1$ x0: 0$ y0: 1$
(%i28) for i:1 thru 3 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 0.1, 1.221551365567124]$

$[i, x1, y1] = [2, 0.2, 1.492920208470706]$

$[i, x1, y1] = [3, 0.3, 1.825519191636627]$

(%i29)

Ex. 3. $y' = f(x,y) = x^2 + y^2 - 3y(x+1),\ y(0) = 2,\ h = 0.05.$

```
(%i29) f(x,y):=x^2+y^2-3*y*(x+1)$
(%i30) h: 0.05$ x0: 0$ y0: 2$
(%i33) for i:1 thru 3 do
         ( x1: x0+h,
               k1: h*f(x0,y0),
               k2: h*f(x0+0.5*h, y0+0.5*k1),
               k3: h*f(x0+0.5*h, y0+0.5*k2),
               k4: h*f(x0+h, y0+k3),
               y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
               x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 0.05, 1.890339767187235]$

$[i, x1, y1] = [2, 0.1, 1.763094323005745]$

$[i, x1, y1] = [3, 0.15, 1.621677082133333]$

(%i34)

Ex. 4. $y' = f(x,y) = (1+x)/(1-y^2),\ y(2) = 3,\ h = 0.1.$

```
(%i35) f(x,y):=(1+x)/(1-y^2)$
(%i36) h: 0.1$ x0: 2$ y0: 3$
(%i39) for i:1 thru 3 do
         ( x1: x0+h,
               k1: h*f(x0,y0),
               k2: h*f(x0+0.5*h, y0+0.5*k1),
               k3: h*f(x0+0.5*h, y0+0.5*k2),
               k4: h*f(x0+h, y0+k3),
               y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
               x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 2.1, 2.961316248057228]$

$[i, x1, y1] = [2, 2.2, 2.920128957785907]$

$[i, x1, y1] = [3, 2.3, 2.876207301364791]$

(%i40)

Ex. 5. $y' = f(x,y) = \sin(xy) - x^2 y,\ y(1) = \pi,\ h = 0.2.$

```
(%i40) f(x,y):=sin(x*y)-x^2*y$
(%i41) h: 0.2$ x0: 1$ y0: 3.1415$
```

```
(%i44) for i:1 thru 3 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1,
           display([i,x1,y1]) )$
```

$[i, x1, y1] = [1, 1.2, 2.475546873156627]$

$[i, x1, y1] = [2, 1.4, 1.825959686942205]$

$[i, x1, y1] = [3, 1.6, 1.282767384068508]$

```
(%i45)
```

3. **Exercise 13**, p. 108. (Bonus point if you also do Euler's semilinear method)

```
(%i21) f(x,y):=7*exp(-3*x)-3*y$ yex(x):=exp(-3*x)*(7*x+6)$
(%i23) h: 0.1$ x0: 0$ y0: 6$
(%i26) for i:1 thru 10 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [1, 0.1, 4.9, 4.96348207856751, 0.01278982729516185]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [2, 0.2, 3.948572754477203, 4.061206107095796, 0.02773396612937181]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [3, 0.3, 3.148169073399861, 3.293214243898853, 0.04404364847131009]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [4, 0.4, 2.488317113198322, 2.650509064827378, 0.06119275492446566]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [5, 0.5, 1.952657927577367, 2.119736521410083, 0.07882045345974094]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [6, 0.6, 1.523051661408057, 1.686048659860183, 0.09667395866596284]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [7, 0.7, 1.181845384740751, 1.334775067957503, 0.114573374112215]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [8, 0.7999999999999999, 0.9130112690956129, 1.052328258157185, 0.1323892882108299]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [9, 0.8999999999999999, 0.7026104556695179, 0.8266278066989223, 0.1500280416704813]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [10, 0.9999999999999999, 0.5388711778864874, 0.6472318887822316, 0.1674217738245641]$

```
(%i27)
```

```
(%i45) f(x,y):=7*exp(-3*x)-3*y$ yex(x):=exp(-3*x)*(7*x+6)$
(%i47) h: 0.05$ x0: 0$ y0: 6$
```

```
(%i50) for i:1 thru 20 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [1, 0.05, 5.45, 5.465495650299117, 0.002835177500922231]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [2, 0.1, 4.93374779174877, 4.96348207856751, 0.005990610290935368]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [3, 0.15, 4.452972000225055, 4.495278468933502, 0.009411312113548191]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [4, 0.2, 4.008196053258918, 4.061206107095796, 0.01305278590620105]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [5, 0.25, 3.59905071790299, 3.660840783742864, 0.01687865424638865]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [6, 0.3, 3.224521403676897, 3.293214243898853, 0.02085890413878164]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [7, 0.35, 2.883142574034572, 2.956973979989263, 0.02496856802066262]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [8, 0.4, 2.57314940011829, 2.650509064827379, 0.02918671953839436]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [9, 0.45, 2.292594964269818, 2.372048384909907, 0.03349569981183482]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [10, 0.4999999999999999, 2.039439810855407, 2.119736521410084, 0.03788051474494664]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [11, 0.5499999999999999, 1.811619395279046, 1.891691599914428, 0.04232836083799484]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [12, 0.6, 1.607093954004454, 1.686048659860183, 0.04682824863564529]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [13, 0.65, 1.423884471781341, 1.500991455237718, 0.05137070113711299]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [14, 0.7000000000000001, 1.260097726069419, 1.334775067957503, 0.0559475103189904]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [15, 0.7500000000000001, 1.11394281704755, 1.185741276320973, 0.06055153911500327]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [16, 0.8000000000000002, 0.983741230870701, 1.052328258157185, 0.0651765592517901]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [17, 0.8500000000000002, 0.8679312382753039, 0.9330759087137797, 0.0698171175893\backslash 6732]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [18, 0.9000000000000002, 0.765070135634412, 0.8266278066989217, 0.0744684252884\backslash 4488]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [19, 0.9500000000000003, 0.6738315447481625, 0.731730659066706, 0.07912626538348347]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [20, 1.0, 0.5930023253421316, 0.6472318887822307, 0.08378691529265074]$

```
(%i51)
(%i39) f(x,y):=7*exp(-3*x)-3*y$ yex(x):=exp(-3*x)*(7*x+6)$
```

```
(%i41) h: 0.025$ x0: 0$ y0: 6$
(%i44) for i:1 thru 40 do
         ( x1: x0+h, y1: y0+h*f(x0,y0), x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [1, 0.025, 5.725, 5.728816028078814, 0.666111123155433 \times 10^{-3}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [2, 0.05, 5.457980110107497, 5.465495650299117, 0.00137508849562596]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [3, 0.07500000000000001, 5.199255497723819, 5.210318327404935, 0.00212325408659353 4]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [4, 0.1, 4.949051673677424, 4.96348207856751, 0.0029073147966821 2]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [5, 0.125, 4.707515986770918, 4.725113791687934, 0.003724313464783166]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [6, 0.15, 4.474727911551519, 4.495278468933502, 0.004571587171741555]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [7, 0.175, 4.250708244718965, 4.273987507550239, 0.005446731603718906]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [8, 0.2, 4.035427315129236, 4.061206107095797, 0.006347570472111613]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [9, 0.225, 3.828812302810998, 3.856859886102185, 0.007272129172297274]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [10, 0.25, 3.630753753706494, 3.660840783742864, 0.008218612011202822]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [11, 0.275, 3.441111368908185, 3.473012315284723, 0.009185382452040789]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [12, 0.3, 3.259719139921437, 3.293214243898853, 0.01017094591992315]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [13, 0.325, 3.086389894881934, 3.121266725735123, 0.01117393478924004]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [14, 0.35, 2.920919314639342, 2.956973979989262, 0.01219309523651995]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [15, 0.3750000000000001, 2.763089472135843, 2.800127530965766, 0.0132272756938139]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [16, 0.4000000000000001, 2.612671943513366, 2.650509064827378, 0.01427541667980528]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [17, 0.4250000000000001, 2.469430534834499, 2.50789293978713, 0.01533654182059927]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [18, 0.4500000000000001, 2.333123664160658, 2.372048384909907, 0.01640974990091836]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [19, 0.4750000000000001, 2.20350643496164, 2.242741419417789, 0.0174942078103392]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [20, 0.5000000000000001, 2.080332433400976, 2.119736521410082, 0.01858914426916312]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [21, 0.5250000000000001, 1.963355278921878, 2.002798072190151, 0.01969384423520057]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [22, 0.5500000000000002, 1.852329954721939, 1.891691599914428, 0.02080764390679184]$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [23, 0.5750000000000002, 1.747013942126425, 1.786184844023306, 0.02192992624920602]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [24, 0.6000000000000002, 1.647168180527201, 1.686048659860182, 0.02306011698156156]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [25, 0.6250000000000002, 1.552557872426438, 1.591057781016132, 0.02419768096989256]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [26, 0.6500000000000002, 1.462953151192318, 1.500991455237717, 0.02534211897920169]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [27, 0.6750000000000003, 1.378129627380534, 1.415633968189478, 0.02649296474349952]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [28, 0.7000000000000003, 1.297868827884864, 1.334775067957502, 0.02764978231808937]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [29, 0.7250000000000003, 1.221958540737771, 1.258210301903708, 0.02881216368288101]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [30, 0.7500000000000003, 1.150193077074822, 1.185741276320973, 0.02997972656939757]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [31, 0.7750000000000004, 1.082373460592537, 1.117175848286674, 0.03115211248749313]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [32, 0.8000000000000004, 1.018307553757082, 1.052328258157184, 0.03232898493068852]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [33, 0.8250000000000004, 0.9578101290509484, 0.9910192102798284, 0.03351002774154392]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [34, 0.8500000000000004, 0.9007028926671565, 0.9330759087137792, 0.03469494362066216]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [35, 0.8750000000000004, 0.8468144672673216, 0.8783320540402981, 0.03588345276481339]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [36, 0.9000000000000005, 0.7959803397032664, 0.8266278066989211, 0.03707529162132005]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [37, 0.9250000000000005, 0.7480427789549776, 0.7778097217036752, 0.03827021174728648]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [38, 0.9500000000000005, 0.7028507289540471, 0.7317306590667058, 0.03946797876351648]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [39, 0.9750000000000005, 0.6602596804355902, 0.6882496737807628, 0.04066837139408831]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [40, 1.0, 0.6201315254876487, 0.6472318887822306, 0.04187118058347113]$$

(%i45)

The final values and relative errors from the three steps size are $y_{h=0.1}(1) = 0.538871$, $\varepsilon_{h=0.1} = 0.167$, $y_{h=0.05}(1) = 0.6059300$, $\varepsilon_{h=0.05} = 0.083$, $y_{h=0.025}(1) = 0.6201315$, $\varepsilon_{h=0.025} = 0.042$. Notice that the method shows first-order convergence, reducing the interval size h by half reduces the relative error by half.

4. Exercises 13, p. 126.

```
(%i63) f(x,y):=exp(-3*x)*(1-4*x+3*x^2-4*x^3)-3*y$
       yex(x):=-exp(-3*x)*(3-x+2*x^2-x^3+x^4)$
(%i65) h: 0.1$ x0: 0$ y0: -3$
(%i68) for i:1 thru 10 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$[i, x1, y1, \text{yE}, \text{err}] = [1, 0.1, -2.162598010632905, -2.162522467992003, 0.349326501899443 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [2, 0.2, -1.577172164112856, -1.577065117479794, 0.678771167250355 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [3, 0.3, -1.163350794222229, -1.163236453483828, 0.982953535016681 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [4, 0.4, -0.8680302936304252, -0.8679212410462013, 1.256480185833875 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [5, 0.5, -0.6555427389005404, -0.6554448454360126, 1.49354236606591 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [6, 0.6, -0.5015353521322574, -0.501450707309005, 1.687998880421825 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [7, 0.7, -0.3891276734724686, -0.3890563182025489, 1.834060175384279 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [8, 0.7999999999999999, -0.3064680184340518, -0.3064089590303197, 1.92746987291\backslash 2144 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [9, 0.8999999999999999, -0.2451534334639788, -0.2451052255131414, 1.96682672662\backslash 0039 \times 10^{-4}]$

$[i, x1, y1, \text{yE}, \text{err}] = [10, 0.9999999999999999, -0.19918719755121, -0.1991482734714558, 1.95452760275\backslash 7375 \times 10^{-4}]$

```
(%i69)
```

```
(%i69) f(x,y):=exp(-3*x)*(1-4*x+3*x^2-4*x^3)-3*y$
       yex(x):=-exp(-3*x)*(3-x+2*x^2-x^3+x^4)$
(%i71) h: 0.05$ x0: 0$ y0: -3$
```

```
(%i74) for i:1 thru 20 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [1, 0.05, -2.543292302947726, -2.543289861263845, 0.960049390203239 \times 10^{-6}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [2, 0.1, -2.162526572313374, -2.162522467992003, 1.897932359992188 \times 10^{-6}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [3, 0.15, -1.844109484541107, -1.844104303185068, 0.280968708231894 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [4, 0.2, -1.577070939293027, -1.577065117479794, 0.369154904771009 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [5, 0.25, -1.352524433802992, -1.352518293590483, 0.453983693840401 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [6, 0.3, -1.163242677781687, -1.163236453483828, 0.535084491228065 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [7, 0.35, -1.003323597091739, -1.003317456031253, 0.61207551493789 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [8, 0.4, -0.867927182473265, -0.8679212410462016, 0.684558319620866 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [9, 0.45, -0.7530678363014294, -0.7530621723953632, 0.752116660984273 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [10, 0.4999999999999999, -0.655450182857431, -0.6554448454360128, 0.814320450505907 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [11, 0.5499999999999999, -0.5723389177532403, -0.5723339342403538, 0.870735175450322 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [12, 0.6, -0.5014553253524618, -0.501450707309005, 0.920936672249077 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [13, 0.65, -0.4408947053237915, -0.4408904528006046, 0.964530567606513 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [14, 0.7000000000000001, -0.3890602133374718, -0.3890563182025488, 1.001175084622411 \times 10^{-5}]$$

$$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [15, 0.7500000000000001, -0.3446096099591903, -0.3446060584307829, 1.030605330501702 \times 10^{-5}]$$

$[i,\ x1,\ y1,\ \mathrm{yE},\ \mathrm{err}] = [16,\ 0.8000000000000002,\ -0.3064121844648638,\ -0.3064089590303196,\ 1.0526567351\backslash 1536 \times 10^{-5}]$

$[i,\ x1,\ y1,\ \mathrm{yE},\ \mathrm{err}] = [17,\ 0.8500000000000002,\ -0.2735137229442882,\ -0.2735108038042017,\ 1.0672851111\backslash 8625 \times 10^{-5}]$

$[i,\ x1,\ y1,\ \mathrm{yE},\ \mathrm{err}] = [18,\ 0.9000000000000002,\ -0.2451078593671625,\ -0.2451052255131413,\ 1.07458093382\backslash 0204 \times 10^{-5}]$

$[i,\ x1,\ y1,\ \mathrm{yE},\ \mathrm{err}] = [19,\ 0.9500000000000003,\ -0.2205125132319408,\ -0.2205101432419989,\ 1.07477592959\backslash 5034 \times 10^{-5}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [20, 1.0, -0.1991504008547125, -0.1991482734714556, 1.068240873886497 \times 10^{-5}]$

```
(%i75)
```

```
(%i75) f(x,y):=exp(-3*x)*(1-4*x+3*x^2-4*x^3)-3*y$
       yex(x):=-exp(-3*x)*(3-x+2*x^2-x^3+x^4)$
(%i77) h: 0.025$ x0: 0$ y0: -3$
(%i80) for i:1 thru 40 do
         ( x1: x0+h,
              k1: h*f(x0,y0),
              k2: h*f(x0+0.5*h, y0+0.5*k1),
              k3: h*f(x0+0.5*h, y0+0.5*k2),
              k4: h*f(x0+h, y0+k3),
              y1: y0+(1/6)*(k1+2*k2+2*k3+k4),
              x0: x1, y0: y1, yE: yex(x1), err: abs((yE-y1)/yE),
           display([i,x1,y1,yE,err]) )$
```

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [1, 0.025, -2.761182495226335, -2.761182417593181, 2.811590919629436 \times 10^{-8}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [2, 0.05, -2.543290003551932, -2.543289861263845, 0.559464687110792 \times 10^{-7}]$

$[i,\ x1,\ y1,\ \mathrm{yE},\ \mathrm{err}] = [3,\ 0.07500000000000001,\ -2.344331834515073,\ -2.344331638854666,\ 0.83461061457\backslash 6172 \times 10^{-7}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [4, 0.1, -2.162522707232189, -2.162522467992003, 1.106301506085183 \times 10^{-7}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [5, 0.125, -1.996260174183906, -1.996259899847704, 1.374250927939822 \times 10^{-7}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [6, 0.15, -1.844104605282431, -1.844104303185069, 1.638179369449098 \times 10^{-7}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [7, 0.175, -1.704761441134865, -1.704761117603225, 1.897812170633416 \times 10^{-7}]$

$[i, x1, y1, \mathrm{yE}, \mathrm{err}] = [8, 0.2, -1.577065457002586, -1.577065117479794, 2.152877439567263 \times 10^{-7}]$

$$[i, x1, y1, \text{yE}, \text{err}] = [9, 0.225, -1.45996680798114, -1.459966457136014, 0.24031040189155 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [10, 0.25, -1.352518651767027, -1.352518293590484, 0.264821958424507 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [11, 0.275, -1.253866168377135, -1.253865806267094, 0.288794893296854 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [12, 0.3, -1.163236816647704, -1.163236453483828, 0.312201250700879 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [13, 0.325, -1.079931685535221, -1.079931323744703, 0.335012522896386 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [14, 0.35, -1.003317814415813, -1.003317456031253, 0.357199566394716 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [15, 0.3750000000000001, -0.9328213709509302, -0.9328210176612347, 0.3787325637\backslash 0667 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [16, 0.4000000000000001, -0.867921587851058, -0.8679212410462014, 0.39958102211\backslash 2287 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [17, 0.4250000000000001, -0.8081453712008229, -0.8081450320111808, 0.419713824\backslash 3813 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [18, 0.4500000000000001, -0.753062503064455, -0.753062172395363, 0.43909932555\backslash 1113 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [19, 0.4750000000000001, -0.7022813700101411, -0.7022810485722395, 0.4577055044\backslash 8811 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [20, 0.5000000000000001, -0.6554451571001412, -0.6554448454360123, 0.47550015994\backslash 2172 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [21, 0.5250000000000001, -0.6122284539023104, -0.6122281524098446, 0.49245116315\backslash 5407 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [22, 0.5500000000000002, -0.572334225287471, -0.5723339342403536, 0.50852675334\backslash 1971 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [23, 0.5750000000000002, -0.5354911052748305, -0.5354908248404925, 0.52369587855\backslash 6236 \times 10^{-6}]$$

$$[i, x1, y1, \text{yE}, \text{err}] = [24, 0.6000000000000002, -0.5014509770536684, -0.5014507073090047, 0.53792857360\backslash 6622 \times 10^{-6}]$$

$[i, x1, y1, yE, err] = [25, 0.6250000000000002, -0.4699868066145442, -0.4699865475596645, 0.55119637171\backslash 5218 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [26, 0.6500000000000002, -0.4408907012303547, -0.4408904528006044, 0.56347273738\backslash 5992 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [27, 0.6750000000000003, -0.4139721673928694, -0.4139719294693267, 0.57473351623\backslash 2095 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [28, 0.7000000000000003, -0.3890565457839152, -0.3890563182025485, 0.58495738546\backslash 9922 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [29, 0.7250000000000003, -0.3659836034867012, -0.3659833860463469, 0.59412629814\backslash 1839 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [30, 0.7500000000000003, -0.3446062659614799, -0.3446060584307828, 0.60222590999\backslash 1902 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [31, 0.7750000000000004, -0.3247894733560878, -0.3247892754795301, 0.60924597156\backslash 9487 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [32, 0.8000000000000004, -0.3064091475271909, -0.3064089590303193, 0.61518067928\backslash 7268 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [33, 0.8250000000000004, -0.28935125774013, -0.2893510783340781, 0.62002897307\backslash 9328 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [34, 0.8500000000000004, -0.2735109744188106, -0.2735108038042016, 0.62379477016\backslash 3831 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [35, 0.8750000000000004, -0.2587919015540582, -0.2587917394243654, 0.62648712512\backslash 3954 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [36, 0.9000000000000005, -0.2451053794687126, -0.2451052255131411, 0.62812031511\backslash 5568 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [37, 0.9250000000000005, -0.2323698505977129, -0.232369704503664, 0.62871383857\backslash 1457 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [38, 0.9500000000000005, -0.2205102817868316, -0.2205101432419988, 0.62829233492\backslash 8207 \times 10^{-6}]$

$[i, x1, y1, yE, err] = [39, 0.9750000000000005, -0.2094576373580673, -0.2094575060522114, 0.62688541645\backslash 1722 \times 10^{-6}]$

$$[i, x1, y1, \text{yE}, \text{err}] = [40, 1.0, -0.1991483978450135, -0.1991482734714556, 0.624527422631212 \times 10^{-6}]$$

(%i81)

The final values and relative errors from the three steps size are $y_{h=0.1}(1) = -0.199187$, $\varepsilon_{h=0.1} = 1.93 \times 10^{-4}$, $y_{h=0.05}(1) = -0.199148273$, $\varepsilon_{h=0.05} = 1.068 \times 10^{-5}$, $y_{h=0.025}(1) = -0.1991483978450$, $\varepsilon_{h=0.025} = 6.24 \times 10^{-6}$. Notice that the method shows high-order convergence (fourth), reducing the interval size h by half reduces the relative error by sixteen.