Homework 12

Due date: April 23, 2020, 11:55PM.

Bibliography: Lessons 22-23

This final homework is a guided tour to some of the behavior encountered in the study of dynamical system.

Problems

Determine the local linearization, eigenvalue maps, and Poincaré sections of the following systems at two different parameter set values of your choice.

1. Duffing oscillator
2. Lorenz system

1 Example: Double pendulum

1.1 Problem definition

Consider system

$$u' = v, v' = -\sin u - a v + b\cos(\omega t)$$

that describes the motion of a forced, damped, planar pendulum

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots -$$

1.2 Problem analysis

This is a inhomogeneous, nonlinear system of two differential equations of first order,

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \text{with } \boldsymbol{y} = \begin{pmatrix} u \\ v \end{pmatrix}, \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} v \\ -\sin u - a v + b\cos(\omega t) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\cos u & -a \end{pmatrix}$$

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ \cos u & \lambda + a \end{vmatrix} = \lambda^2 + a\lambda + \cos u = 0 \Rightarrow \lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4\cos u}}{2}$$

Observations.

- If $a = 0$, then $\lambda_{1,2} = \pm i\sqrt{\cos u}$, locally, around (u_k, v_k) the system behaves as $\cos((\sqrt{\cos u_k})t)$,$\sin((\sqrt{\cos u_k})t)$
- If $0 < a < 2$ then the system has complex-conjugate roots and the system spirals towards origin
- If $-2 < a < 0$ then the system has complex-conjugate roots and the system spirals out to infinity
- If $a \geqslant 2$, the system decays towards origin
- If $a \leqslant -2$, the system evolves towards infinity

1.3 System trajectory

1.3.1 Trajectory plot

```
(%i3) plotdf([v, -sin(u)+0.0*v], [u,v], [trajectory_at,0.7,0.7],[direction,forward],
          [u,-1,1],[v,-1,1],[tstep,0.01],[nsteps,1000])$
(%i4)
```

1.3.2 Numerical compuation of trajectory

```
(%i81) (Nsteps: 31, Ncycles: 100, a: 0.2, b: 1.0, w: 2.0)$
(%i82) [dudt: v, dvdt: -sin(u)-a*v+b*cos(w*t), T: 2*3.14159/w ];
```

(%o82) $[v, -0.2\,v - \sin(u) + 1.0\cos(2.0\,t), 3.14159]$

```
(%i83) [dt: T/Nsteps, tmax: Ncycles*T ];
```

(%o83) $[0.1013416129032258, 314.159]$

```
(%i84) tuvL : rk( [dudt,dvdt], [u,v], [0.8,0.8], [t,0,tmax,dt] )$
(%i85) N: length(tuvL)$
(%i86)
(%i69)

(%i86) tuL: makelist( [tuvL[i][1], tuvL[i][2]],i,1,N)$
(%i87) tvL: makelist( [tuvL[i][1], tuvL[i][3]],i,1,N)$
(%i89) plot2d( [ [discrete,tuL], [discrete,tvL]], [x,0,tmax],
                 [legend, "u", "v"], [xlabel, "t"])$
(%i90)
```

1.3.3 Numerical computation of phase space plot

```
(%i90) uvL: makelist( [tuvL[i][2], tuvL[i][3]],i,1,N)$
(%i91) plot2d( [ [discrete,uvL] ], [xlabel, "u"], [ylabel, "v"] )$
(%i92)
```

1.3.4 Numerical computation of Poincaré section plot

```
(%i92) Nstart: 50*Nsteps$
(%i93) PoincareL: makelist( uvL[Nsteps*i+Nstart], i, 0, floor((N-Nstart)/Nsteps) )$
(%i94) plot2d( [discrete,PoincareL], [x,-50,50], [y,-50,50], [style,[points,1,1,1]] )$
(%i95)
(%i31)
```

2 Example: Duffing oscillator

2.1 Problem definition

Consider system $u'' + \delta u' + \alpha u + \beta u^3 = \gamma \sin(\omega t)$

$$u' = v, v' = -\delta v - \alpha u - \beta u^3 + \gamma \sin(\omega t)$$

that describes a nonlinear, forced oscillator.

2.2 Problem analysis

This is a inhomogeneous, nonlinear system of two differential equations of first order,

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}), \text{with } \boldsymbol{y} = \begin{pmatrix} u \\ v \end{pmatrix}, \boldsymbol{f}(\boldsymbol{y}) = \begin{pmatrix} v \\ -\delta v - \alpha u - \beta u^3 + \gamma \sin(\omega t) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha - 3\beta u^2 & -\delta \end{pmatrix}$$

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda & -1 \\ \alpha + 3\beta u^2 & \lambda + \delta \end{vmatrix} = \lambda^2 + \delta\lambda + (\alpha + 3\beta u^2) = 0 \Rightarrow \lambda_{1,2} = \frac{-\delta \pm \sqrt{\delta^2 - 4(\alpha + 3\beta u^2)}}{2}$$

Observations.

- If $\delta=0$, then $\lambda_{1,2}=\pm\frac{i}{2}\sqrt{\alpha+3\beta u^2}$, locally, around $(u_k,\ v_k)$ the system behaves as $\cos\left(\left(\frac{1}{2}\sqrt{\alpha+3\beta u^2}\right)t\right)$,$\sin\left(\left(\frac{1}{2}\sqrt{\alpha+3\beta u^2}\right)t\right)$
- If $\delta>0$ the system will decay
- If $\delta<0$ the system amplitude will grow exponentially, without bound

2.3 System trajectory

2.3.1 Trajectory plot

```
(%i95) plotdf([v, -0.2*v-0.5*u-0.1*u^3], [u,v], [trajectory_at,0.7,0.7],[direction,
       forward],
               [u,-1,1],[v,-1,1],[tstep,0.01],[nsteps,1000])$
(%i96)
```

2.3.2 Numerical compuation of trajectory

```
(%i44) (Nsteps: 31, Ncycles: 100, a: 1, b: 0.2, d:0.1 , g:1, w: 1.1)$
(%i45) [dudt: v, dvdt: -d*v-a*u-b*u^3+g*sin(w*t) , T: 2*3.14159/w ];
```

(%o45) $[v,-0.1\,v-0.2\,u^3-u+\sin(1.1\,t),5.711981818181817]$

```
(%i46) [dt: T/Nsteps, tmax: Ncycles*T ];
```

(%o46) $[0.1842574780058651,571.1981818181818]$

```
(%i47) tuvL : rk( [dudt,dvdt], [u,v], [0.1,0.5], [t,0,tmax,dt] )$
(%i48) N: length(tuvL)$
(%i49)
(%i69)

(%i49) tuL: makelist( [tuvL[i][1], tuvL[i][2]],i,1,N)$
(%i50) tvL: makelist( [tuvL[i][1], tuvL[i][3]],i,1,N)$
(%i51) plot2d( [ [discrete,tuL], [discrete,tvL]], [x,0,tmax],
               [legend, "u", "v"], [xlabel, "t"])$
(%i52)
```

2.3.3 Numerical computation of phase space plot

```
(%i52) uvL: makelist( [tuvL[i][2], tuvL[i][3]],i,1,N)$
(%i53) plot2d( [ [discrete,uvL] ], [xlabel, "u"], [ylabel, "v"] )$
(%i54)
```

2.3.4 Numerical computation of Poincaré section plot

```
(%i54) Nstart: 1*Nsteps$
(%i55) PoincareL: makelist( uvL[Nsteps*i+Nstart], i, 0, floor((N-Nstart)/Nsteps) )$
(%i58) plot2d( [discrete,PoincareL], [style,[points,1,1,1]] )$
(%i59)

(%i31)
```

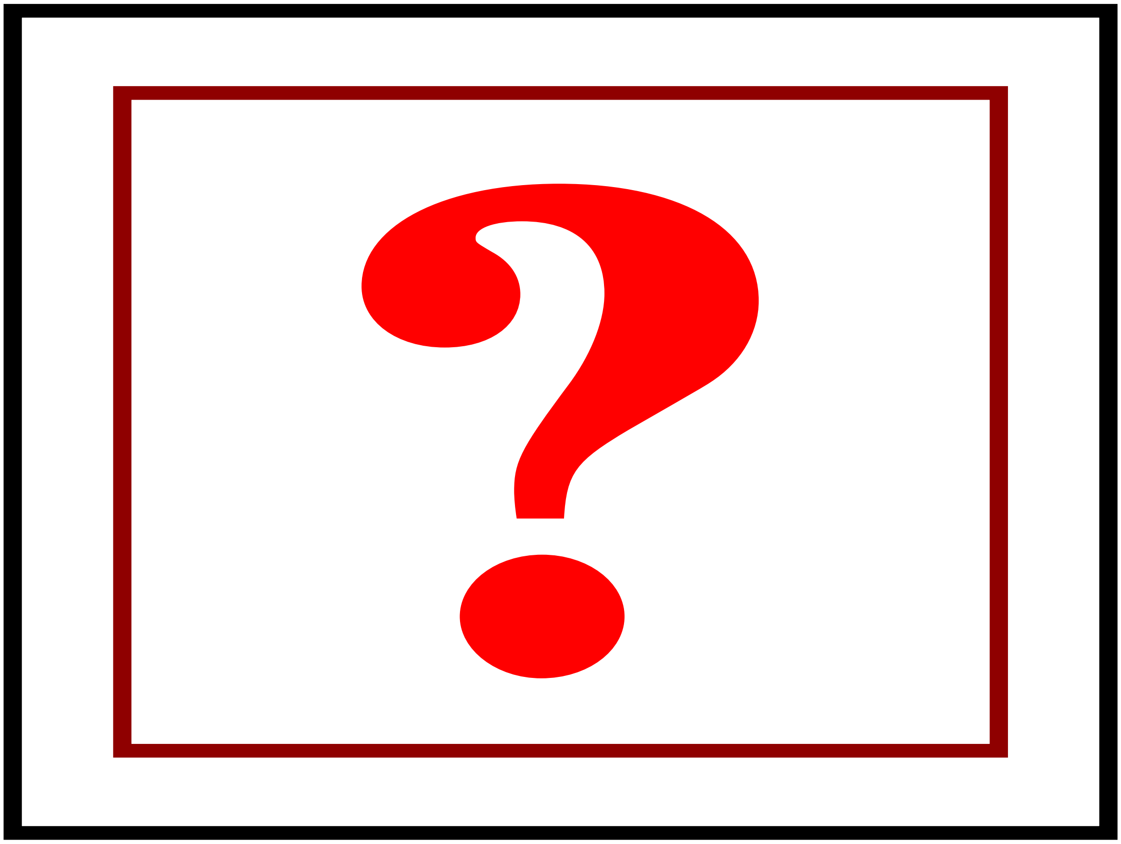

Figure 1. Poincaré section for Duffing oscillator

3 Example: Lorenz system

3.1 Problem definition

Consider system

$$\begin{aligned} x' &= \sigma\,(y-x)\\ y' &= x(\rho-z)-y\\ z' &= xy-\beta z \end{aligned}$$

that describes atmospheric convection (e.g., cumulo-nimbus clouds).

3.2 Problem analysis

This is a homogeneous, nonlinear system of three differential equations of first order,

$$\boldsymbol{u}' = \boldsymbol{f}(\boldsymbol{u}), \text{with } \boldsymbol{u} = \begin{pmatrix} x\\ y\\ z \end{pmatrix}, \boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} \sigma\,(y-x)\\ x(\rho-z)-y\\ xy-\beta z \end{pmatrix} = \begin{pmatrix} f_1\\ f_2\\ f_3 \end{pmatrix}$$

$$\boldsymbol{A} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0\\ \rho-z & -1 & -x\\ y & x & -\beta \end{pmatrix}$$

Characteristic polynomial

$$p_{\boldsymbol{A}}(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A})$$

is a cubic, with complicated analytical form of the roots. Analyze behavior around equilibria

- At $x=y=z=0$

  $$\boldsymbol{A}=\begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}, p_{\boldsymbol{A}}(\lambda)=\det(\lambda\boldsymbol{I}-\boldsymbol{A})=\begin{vmatrix} \lambda+\sigma & -\sigma & 0 \\ -\rho & \lambda+1 & 0 \\ 0 & 0 & \lambda+\beta \end{vmatrix}=(\lambda+\beta)[\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho)]$$

  has eigenvalues $\lambda_1=-\beta$ and solutions of quadratic equation $\lambda^2+(\sigma+1)\lambda+\sigma(1-\rho)=0$

  $$\lambda_{2,3}=\frac{-1-\sigma\pm\sqrt{(\sigma+1)^2-4\sigma(1-\rho)}}{2}$$

  - If $\lambda_1,\lambda_2,\lambda_3<0$ the equilibrium point is stable
  - If any of $\lambda_1,\lambda_2,\lambda_3$ is positive, equilibrium is unstable.

3.3 System trajectory

3.3.1 Trajectory plot

```
(%i60) plotdf([s*(y-x),x*(r-z)-y], [x,y], [trajectory_at,0.7,0.7],[direction,forward],
          [x,-1,1],[y,-1,1],[tstep,0.01],[nsteps,1000],
          [parameters,"s=0.2,r=0.3,z=0"],
          [sliders,"s=0.1:0.3,r=0.1:0.5,z=-1:1"]
         )$
(%i62)
```

3.3.2 Numerical computation of trajectory

$$\boldsymbol{u}'=\boldsymbol{f}(\boldsymbol{u}), \text{with } \boldsymbol{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \boldsymbol{f}(\boldsymbol{u})=\begin{pmatrix} \sigma(y-x) \\ x(\rho-z)-y \\ xy-\beta z \end{pmatrix}=\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

```
(%i75) (Nsteps: 31, Ncycles: 100, s: 0.1, r: 0.2, b:0.1)$
(%i76) [dxdt: s*(y-x), dydt: x*(r-z)-y, dzdt: x*y-b*z  , T: 3.1415 ];
```

(%o76) $[0.1\,(y-x), x\,(0.2-z)-y, x\,y-0.1\,z, 3.1415]$

```
(%i77) [dt: T/Nsteps, tmax: Ncycles*T ];
```

(%o77) $[0.1013387096774194, 314.15]$

```
(%i78) txyzL : rk( [dxdt,dydt,dzdt], [x,y,z], [0.1,0.5,0.2], [t,0,tmax,dt] )$
(%i79) N: length(txyzL);
```

(%o79) 3102

```
(%i80)
(%i69)

(%i80) txL: makelist( [txyzL[i][1], txyzL[i][2]],i,1,N)$
(%i81) tyL: makelist( [txyzL[i][1], txyzL[i][3]],i,1,N)$
(%i82) tzL: makelist( [txyzL[i][1], txyzL[i][4]],i,1,N)$
(%i83) plot2d( [ [discrete,txL], [discrete,tyL], [discrete,tzL] ],
          [legend, "x", "y", "z"], [xlabel, "t"])$
(%i84)
```

3.3.3 Numerical computation of phase space plot

```
(%i52) uvL: makelist( [tuvL[i][2], tuvL[i][3]],i,1,N)$
(%i53) plot2d( [ [discrete,uvL] ], [xlabel, "u"], [ylabel, "v"] )$
(%i54)
```

3.3.4 Numerical computation of Poincaré section plot

```
(%i54) Nstart: 1*Nsteps$
(%i55) PoincareL: makelist( uvL[Nsteps*i+Nstart], i, 0, floor((N-Nstart)/Nsteps) )$
(%i58) plot2d( [discrete,PoincareL], [style,[points,1,1,1]] )$
(%i59)
(%i31)
```

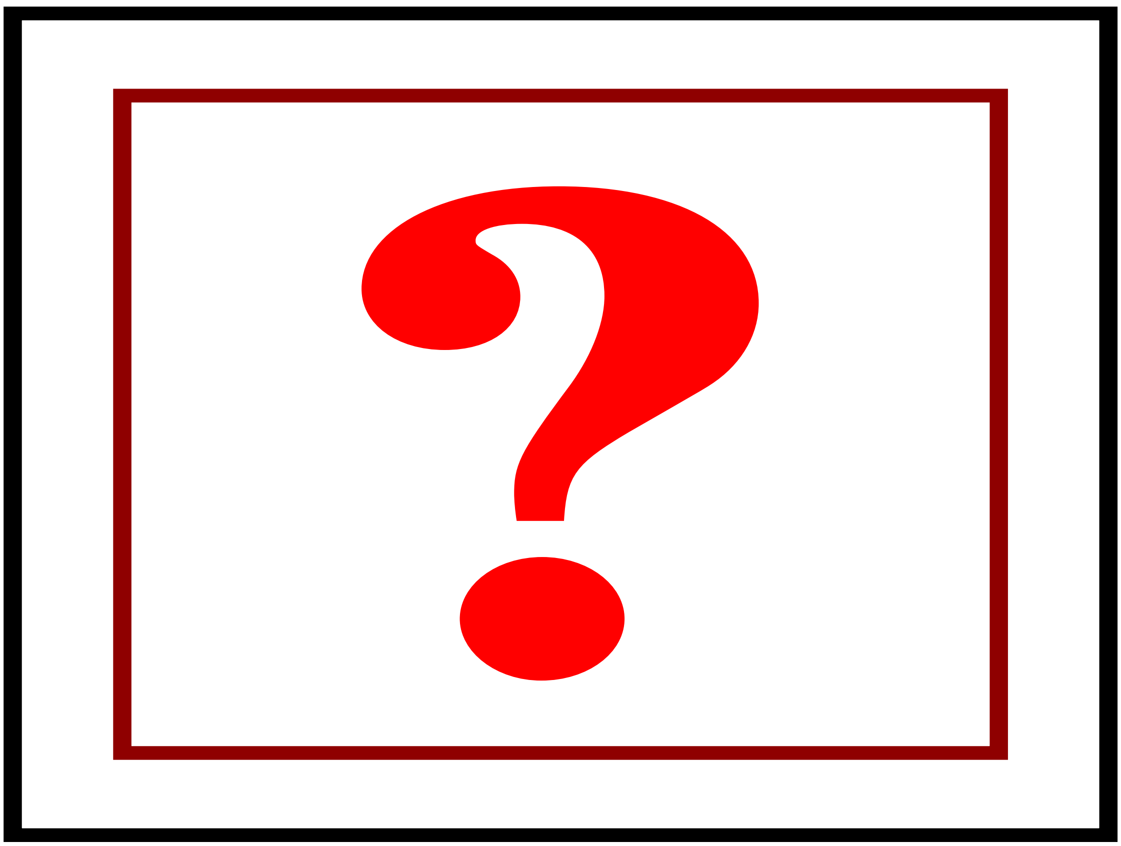

Figure 2. Poincaré section for Duffing oscillator