Test 2 Solution

Solve the following problems (3 course points each). Present a brief motivation of your method of solution. Explicitly state any conditions that must be met for solution procedure to be valid. Sketch out a solution for yourself on scratch paper, and then neatly transcribe so the solution you present is readily legible.

No credit is awarded for statement of the final answer to a problem without presentation of the solution procedure.

1. At $t=0$ an object is placed in a room with temperature of 20° C. The temperature of the object drops by 5° C in 4 minutes and by 7° C in 8 minutes. What was the temperature of the object at $t=0$?

   Solution. The initial temperature is $T(0)=T_0$, and the rate of change of the object temperature T' is proportional to the difference with respect to ambient temperature $T_m=20°$ C,

$$T' = -k(T - T_m), T(0) = T_0.$$

The solution to the above initial value problem (IVP) is $T(t)=T_m+(T_0-T_m)e^{-kt}$, which evaluated at $t=4$ and $t=8$ gives

$$T(4) = 20 + (T_0 - 20)e^{-4k}, T(8) = 20 + (T_0 - 20)e^{-8k}.$$

From problem information $T_0 - T(4) = 5, T_0 - T(8) = 7$, or $T(4) = T_0 - 5$, $T(8) = T_0 - 7$, which leads to

$$\left\{ \begin{array}{l} T_0 - 5 = 20 + (T_0 - 20)e^{-4k} \\ T_0 - 7 = 20 + (T_0 - 20)e^{-8k} \end{array} \right. \Rightarrow \left\{ \begin{array}{l} e^{-4k} = \dfrac{T_0 - 25}{T_0 - 20} \\ e^{-8k} = \dfrac{T_0 - 27}{T_0 - 20} \end{array} \right. .$$

Recall that $(e^{-4k})^2 = e^{-4k} \cdot e^{-4k} = e^{-8k}$ to obtain

$$\left(\frac{T_0 - 25}{T_0 - 20} \right)^2 = \frac{T_0 - 27}{T_0 - 20} \Rightarrow (25 - T_0)^2 = (27 - T_0)(20 - T_0) \Rightarrow 625 - 50T_0 + T_0^2 = 540 - 47T_0 + T_0^2 \Rightarrow 3T_0 = 85,$$

and the initial temperature is $T_0 = 85/3 \cong 28.3°$ C.

2. Determine whether $\boldsymbol{B}$ is a basis set for real-valued 2 by 2 matrices

$$\boldsymbol{B} = \{ \ \boldsymbol{B}_1, \ \boldsymbol{B}_2, \ \boldsymbol{B}_3 \ \} = \left\{ \left(\begin{array}{cc} 1 & 3 \\ 2 & 1 \end{array} \right), \left(\begin{array}{cc} -1 & 2 \\ 1 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & 1 \\ 0 & -4 \end{array} \right) \right\}.$$

Solution. Note that a 2 by 2 matrix can also be represented as a 4 component vector (HW06 2.Ex2 solution)

$$\boldsymbol{A} = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \rightarrow \boldsymbol{a} = (\ a_{11} \ \ a_{12} \ \ a_{21} \ \ a_{22} \ ), \boldsymbol{a} \in \mathbb{R}^4.$$

Recall that the dimension of a vector space is the number of vectors in a basis, and the fact that the dimension of $\mathbb{R}^4$ is 4, while $\boldsymbol{B}$ only contains 3 elements implies that $\boldsymbol{B}$ is not a basis set (this observation is sufficient to solve the problem). To obtain a counterexample (optional), try to obtain some general $\boldsymbol{A}$ by a linear combination of elements of $\boldsymbol{B}$

$$c_1\boldsymbol{B}_1 + c_2\boldsymbol{B}_2 + c_3\boldsymbol{B}_3 = \left(\begin{array}{cc} c_1 - c_2 & 3c_1 + 2c_2 + c_3 \\ 2c_1 + c_2 & c_1 - 4c_3 \end{array} \right) = \left(\begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right),$$

leading to a system of 4 equations with 3 unknowns

$$\left\{\begin{array}{llll} c_1 & -c_2 & & = a_{11} \\ 3c_1 & +2c_2 & +c_3 & = a_{12} \\ 2c_1 & +c_2 & & = a_{21} \\ c_1 & & -4c_3 & = a_{22} \end{array}\right. .$$

Try to find a simple counterexample: set $a_{11}=0, a_{21}=3$ to obtain $c_1=c_2=1$, leading to the equations

$$\left\{\begin{array}{lll} c_3 & = & a_{12}-5 \\ -4c_3 & = & a_{22}-1 \end{array}\right. ,$$

that cannot be both true, for example for $a_{22}=1$, and $a_{12}=6$ the contradictory conditions $c_3=1$ (first equation), $c_3=0$ (second equation) are obtained, and $\boldsymbol{B}$ is not a basis since it does not span the set of 2 by 2 matrices, in particular the matrix

$$\boldsymbol{A}=\left(\begin{array}{cc} 0 & 6 \\ 3 & 1 \end{array}\right),$$

cannot be represented by a linear combination of elements of $\boldsymbol{B}$.

3. Let

$$\mathcal{S}=\left\{\left(\begin{array}{c} 2s-t \\ s \\ t \\ -s \end{array}\right), s,t\in\mathbb{R}\right\}.$$

a) Prove that $(\mathcal{S},+,\mathbb{R},\cdot)$ a subspace of $(\mathbb{R}^4,+,\mathbb{R},\cdot)$.

Solution. Note that for $s=t=0$, the vector $\mathbf{0}=(0,0,0,0)$ is verified to be an element of $\mathcal{S}$. Verify closure, $c_1,c_2\in\mathbb{R}$, $\boldsymbol{u}_1,\boldsymbol{u}_2\in\mathcal{S}$

$$\boldsymbol{u}_1=\left(\begin{array}{c} 2s_1-t_1 \\ s_1 \\ t_1 \\ -s_1 \end{array}\right), \boldsymbol{u}_2=\left(\begin{array}{c} 2s_2-t_2 \\ s_2 \\ t_2 \\ -s_2 \end{array}\right), c_1\boldsymbol{u}_1+c_2\boldsymbol{u}_2=\left(\begin{array}{c} 2(c_1s_1+c_2s_2)-(c_1t_1+c_2t_2) \\ c_1s_1+c_2s_2 \\ c_1t_1+c_2t_2 \\ -c_1s_1-c_2s_2 \end{array}\right)=\left(\begin{array}{c} 2s-t \\ s \\ t \\ -s \end{array}\right)\in\mathcal{S}$$

with $s=c_1s_1+c_2s_2$, $t=c_1t_1+c_2t_2$, so $(\mathcal{S},+,\mathbb{R},\cdot)$ is indeed a subspace of $(\mathbb{R}^4,+,\mathbb{R},\cdot)$.

b) Find two vectors that span $\mathcal{S}$.

Solution. Write

$$\left(\begin{array}{c} 2s-t \\ s \\ t \\ -s \end{array}\right)=s\left(\begin{array}{c} 2 \\ 1 \\ 0 \\ -1 \end{array}\right)+t\left(\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array}\right)$$

and obtain that the vectors

$$\left\{\left(\begin{array}{c} 2 \\ 1 \\ 0 \\ -1 \end{array}\right),\left(\begin{array}{c} -1 \\ 0 \\ 1 \\ 0 \end{array}\right)\right\}$$

span $\mathcal{S}$.

4. Find and subsequently sketch the solution to the initial value problem

$$y''-14y'+49y=0, y(1)=2, y'(1)=11.$$

Solution. Try $y=e^{rt}$ in the above homogeneous second-order differential equation to obtain

$$r^2-14r+49=(r-7)^2=0$$

with double root $r=7$. Two independent solutions are $y_1=e^{7t}$, $y_2=t\,e^{7t}$, with derivatives $y_1'=7\,e^{7t}$, $y_2'=(1+7t)e^{7t}$. Impose initial conditions on linear combination $y=c_1y_1+c_2y_2$

$$\begin{cases} c_1y_1(1)+c_2y_2(1)=y(1) \\ c_1y_1'(1)+c_2y_2'(1)=y'(1) \end{cases} \Rightarrow \begin{cases} c_1e^7+c_2e^7=2 \\ 7c_1e^7+8c_2e^7=11 \end{cases} \Rightarrow \begin{cases} c_1+c_2=2e^{-7} \\ 7c_1+8c_2=11e^{-7} \end{cases} \Rightarrow \begin{cases} c_1=5e^{-7} \\ c_2=-3e^{-7} \end{cases},$$

and the solution is

$$y(x)=(5-3t)e^{7(t-1)}.$$