Test 3 - Solution

1. Transform the following system of first-order differential equations

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = f(t, y_1, y_4, y_5, y_6) \\ y_4' = y_5 \\ y_5' = y_4 \\ y_6' = g(t, y_4, y_5, y_6) \end{cases},$$

into a system of two third-order equations for dependent variables $u(t), v(t)$.

Solution. Take eq 5 to be $y_5' = y_6$, $y_1 = u$, $u' = y_1' = y_2$, $u'' = y_2' = y_3$, $u''' = y_3' = f(t, u, v, v', v'')$. Take $y_4 = v$, $v' = y_4' = y_5$, $y_5' = y_6 = v''$, $y_6' = v''' = g(t, v, v', v'')$. Assuming correction of 5th equation

$$u''' = f(t, u, v, v', v''), v''' = g(t, v, v', v'').$$

If $y_4' = y_5$. $y_1 = u$, $u' = y_1' = y_2$, $u'' = y_2' = y_3$, $u''' = y_3' = f(t, u,)$. Take $y_4 = v$, $v' = y_4' = y_5$. Would need to introduce additional variable, w, cannot be put into form of two third order equations.

2. Consider the system of differential equations $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$,

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} a & b \\ -b & 3a \end{pmatrix}.$$

Determine a, b for the fundamental set of solutions to be:

a) $\{e^{2t}, te^{2t}\}$

b) $\{e^{-6t}\cos 4t, e^{-6t}\sin 4t\}$

c) $\{e^{6t}, e^{14t}\}$

Solution. Determine roots of characteristic polynomial

$$P(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - a & b \\ -b & \lambda - 3a \end{vmatrix} = \lambda^2 - 4a\lambda + 3a^2 + b^2.$$

Roots are

$$\lambda_{1,2} = 2a + \sqrt{4a^2 - (3a^2 + b^2)} = 2a + \sqrt{a^2 - b^2}.$$

(a) The set $\{e^{2t}, te^{2t}\}$ indicates a double root $\lambda_1 = \lambda_2$, $a = \pm b$, $a = 1$

(b) The set $\{e^{-6t}\cos 4t, e^{-6t}\sin 4t\}$ indicates complex conjugate pair $\lambda_{1,2} = \alpha \pm i\beta$, with $\alpha = -6$, $\beta = 4$. Complex conjugate roots arise if $a^2 - b^2 < 0$, $2a = -6 \Rightarrow a = -3$, $\sqrt{a^2 - b^2} = \sqrt{(-1)(b^2 - a^2)} = i\sqrt{b^2 - a^2}$, $\sqrt{b^2 - a^2} = 4 \Rightarrow b^2 - 9 = 16 \Rightarrow b = \pm 5$.

(c) The set $\{e^{6t}, e^{14t}\}$ indicates distinct roots

$$\begin{matrix} 2a + \sqrt{a^2 - b^2} = 14 \\ 2a - \sqrt{a^2 - b^2} = 6 \end{matrix} \Rightarrow a = 5, 10 + \sqrt{25 - b^2} = 14 \Rightarrow 25 - b^2 = 16 \Rightarrow b = \pm 3.$$

3. Complete the set $\{e^{(p+q)t}\}$ to form a fundamental set of solutions for the system of differential equations $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} p & q & 1 \\ q & p & -1 \\ 1 & -1 & 1 \end{pmatrix}.$$

Solution. The system $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$ has 3 fundamental solutions, of which one is $e^{(p+q)t}$. The characteristic polynomial is

$$P(\lambda) = \det(\lambda \boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - p & -q & -1 \\ -q & \lambda - p & 1 \\ -1 & 1 & \lambda - 1 \end{vmatrix},$$

and is of degree 3. We must determine two more eigenvalues. Since $p+q$ is a root, the polynomial can be written as

$$P(\lambda) = (\lambda - p - q)Q(\lambda) = \begin{vmatrix} \lambda - p & -q & -1 \\ -q & \lambda - p & 1 \\ -1 & 1 & \lambda - 1 \end{vmatrix},$$

with $Q(\lambda)$ a polynomial of degree 2, $Q(\lambda) = \lambda^2 + a\lambda + b$,

$$(\lambda - p - q)(\lambda^2 + a\lambda + b) = \begin{vmatrix} \lambda - p & -q & -1 \\ -q & \lambda - p & 1 \\ -1 & 1 & \lambda - 1 \end{vmatrix}.$$

Take $\lambda = 0$

$$-(p+q)b = \begin{vmatrix} -p & -q & -1 \\ -q & -p & 1 \\ -1 & 1 & -1 \end{vmatrix} = (-p)\begin{vmatrix} -p & 1 \\ 1 & -1 \end{vmatrix} + q\begin{vmatrix} -q & 1 \\ -1 & -1 \end{vmatrix} - \begin{vmatrix} -q & -p \\ -1 & 1 \end{vmatrix} \Rightarrow$$

$$-(p+q)b = -p(p-1) + q(q+1) - (-q-p) = 2(p+q) + q^2 - p^2 \Rightarrow$$

Assume $p + q \neq 0$ (otherwise $e^{(p+q)t}$ is not a non-trivial solution)

$$b = -2 + p - q.$$

Take $\lambda = p$

$$(-q)(p^2 + ap - 2 + p - q) = \begin{vmatrix} 0 & -q & -1 \\ -q & 0 & 1 \\ -1 & 1 & p-1 \end{vmatrix}$$

$$\begin{vmatrix} 0 & -q & -1 \\ -q & 0 & 1 \\ -1 & 1 & p-1 \end{vmatrix} = q\begin{vmatrix} -q & 1 \\ -1 & p-1 \end{vmatrix} - \begin{vmatrix} -q & 0 \\ -1 & 1 \end{vmatrix} = q[q(1-p)+1] + q \Rightarrow$$

$$(-q)(p^2 + ap - 2 + p - q) = q[q(1-p)+1] + q$$

Assume $q \neq 0$,

$$p^2 + ap - 2 + p - q = -[q(1-p) + 1 + 1] = -(q - pq + 2) = pq - q - 2 \Rightarrow$$

$$p^2 + ap + p = pq$$

Assume $p \neq 0$,

$$p + a + 1 = q \Rightarrow a = q - p - 1$$

Roots of $Q(\lambda)$ are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2},$$

and the completion of the fundamental set is $e^{\lambda_1 t}, e^{\lambda_2 t}$.

4. Find a fundamental set of solutions for the system of differential equations $\boldsymbol{y}' = \boldsymbol{A}\boldsymbol{y}$,

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \boldsymbol{A} = \begin{pmatrix} 1 & -q & -1 \\ q & 1 & p \\ 1 & -p & 1 \end{pmatrix},$$

knowing that as $t \to \infty$, $e^{-t}\boldsymbol{y}(t)$ remains finite and non-zero.

Solution. The system has 3 fundamental solutions from which the general solution is

$$\boldsymbol{y} = c_1\boldsymbol{y}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3 = e^{\lambda_1 t}\boldsymbol{k}_1 + c_2\boldsymbol{y}_2 + c_3\boldsymbol{y}_3.$$

Multiply by e^{-t}

$$e^{-t}\boldsymbol{y}(t) = e^{(\lambda_1 - 1)t}\boldsymbol{k}_1 + e^{-t}c_2\boldsymbol{y}_2 + e^{-t}c_3\boldsymbol{y}_3,$$

and $\lambda_1 = 1$, otherwise if $\lambda_1 > 1$, $e^{-t}\boldsymbol{y}(t) \to \infty$, or if $\lambda_1 < 1$ $e^{-t}\boldsymbol{y}(t) \to 0$. Characteristic polynomial is

$$P(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} \lambda - 1 & q & 1 \\ -q & \lambda - 1 & -p \\ -1 & p & \lambda - 1 \end{vmatrix}.$$

Indeed

$$P(\lambda = 1) = 0 = \begin{vmatrix} 0 & q & 1 \\ -q & 0 & -p \\ -1 & p & 0 \end{vmatrix}.$$

Define $Q(\lambda) = \lambda^2 + a\lambda + b$, through

$$P(\lambda) = (\lambda - 1)Q(\lambda) = \begin{vmatrix} \lambda - 1 & q & 1 \\ -q & \lambda - 1 & -p \\ -1 & p & \lambda - 1 \end{vmatrix}$$

$$(\lambda - 1)(\lambda^2 + a\lambda + b) = \begin{vmatrix} \lambda - 1 & q & 1 \\ -q & \lambda - 1 & -p \\ -1 & p & \lambda - 1 \end{vmatrix}$$

At $\lambda = 0$

$$-b = \begin{vmatrix} -1 & q & 1 \\ -q & -1 & -p \\ -1 & p & -1 \end{vmatrix} \Rightarrow b = 2 + p^2 + q^2$$

At $\lambda = 2$

$$4 + 2a + b = \begin{vmatrix} 1 & q & 1 \\ -q & 1 & -p \\ -1 & p & 1 \end{vmatrix} = 2 + p^2 + q^2 \Rightarrow a = -2$$

Roots of $Q(\lambda)$ are

$$\lambda_{2,3} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{2 \pm 2\sqrt{1 - (2 + p^2 + q^2)}}{2} = 1 \pm i\sqrt{1 + p^2 + q^2},$$

and the completion of the fundamental set is $e^{\lambda_1 t}, e^{\lambda_2 t}$.